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SUMMARY

The turbulent flow in a conical diffuser represents the type of turbulent boundary layer with positive longitudinal pressure gradient. In contrast to the boundary layer problem, however, it is not necessary that the pressure distribution along the limits of the boundary layer (along the axis of the diffuser) be given, since this distribution can be obtained from the computation. This circumstance, together with the greater simplicity of the problem as a whole, provides a useful basis for the study of the extension of the results of semiempirical theories to the case of motion with a positive pressure gradient.

In the first part of the paper, formulas are derived for the computation of the velocity and pressure distributions in the turbulent flow along, and at right angles to, the axis of a diffuser of small cone angle. The problem is solved on the basis of the following assumptions:

1. The motion is assumed to take place along straight lines intersecting at the vertex of the diffuser cone.

2. The normal components of the turbulent stress tensor are assumed isotropic. Their gradients along the diffuser are neglected by comparison with the gradient of the static pressure.

3. In the equations of motion the hypothesis of the mixing length (Prandtl formula) is applied, and it is assumed that the curve of dependence of the nondimensional mixing length on the distance from the wall is absolute.

4. In determining the shape of this curve none of the existing turbulence theories is taken as a basis, the assumption, common to all turbulence theories, only being made that the first derivative of the

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mixing length with respect to the distance from the wall is the nondimensional universal turbulence constant χ. The shape of the mixing length curve is chosen as a cubic parabola from the boundary conditions applicable to it.

5. To obtain the solution of the equation of motion in finite form there is applied the step-by-step interpolation of the values of the inertia integral, the applicability of which is proved.

6. At the walls of the diffuser the existence of a laminar sublayer is assumed, the thickness of which follows the known Kármán law. It is shown that the assumption of radial motion in this sublayer is fundamentally inapplicable. In obtaining the velocity distribution in the sublayer, the assumption is made of the continuity of the curve of friction distribution in passing through the boundary of the sublayer.

7. The resistance formula obtained for the diffuser is found to be absolutely identical with that for the pipe.

In the second part of the paper tests are described on the measurement of the velocity and pressure distribution in two conical diffusers with angles of 1° and 2°, and a detailed comparison is made between the experimental and the theoretical results. It is found that:

1. The assumption of the radial character of the flow is satisfied only with a certain, though large, degree of approximation as should be the case if viscous friction exists simultaneously with turbulent friction.

2. The obtained formula for the velocity distribution agrees well with the experimental results.

3. The increase computed (on the basis of the derived resistance formula with the values of the universal constants taken from tests on pipes) in static pressure along the diffuser deviates little from the test results.

In the third part of the paper a first, very approximate, attempt is made at estimating semiempirically the deviation of the true motion from the radial pattern assumed. The analysis is based on the assumption that the true motion may to a first approximation be assumed radial but emanating from another fictitious source. The latter, for the region near the diffuser axis, is computed by two different methods. The good agreement of the results of these computations shows that the assumption is justified. The assumption of the displaced source in computing the characteristics of the growth of the static pressure along the diffuser gives complete agreement

with the experiment for the 1° diffuser, while for the 2° diffuser the disagreement is greater than without the assumption. The result for the 2° diffuser evidently is explained by the fact that the gradient of the normal component of the turbulent stresses was not taken into account together with the pressure gradient. This correction, which is negligible for the 1° diffuser, may have an appreciable effect in the case of the 2° diffuser.

INTRODUCTION

The chief experimental source on which at the present time the so-called semiempirical theories of turbulence are based is the fully developed turbulent flow between two parallel planes or in a straight cylindrical pipe. The reason for such an exclusive role played by these two types of flow lies in the circumstance that these are the simplest types of flow as regards their kinetic and dynamic relations. In the first place, both for the case of flow between parallel walls and for the circular pipe there is no need to consider the change in the mean velocity and friction profiles along the axis of the flow since neither the velocities nor the frictional stresses along the flow direction change in value. In the second place, because of the absence of inertia forces the profile of the shear stresses transverse to the flow direction is found to be linear. This fact is not a consequence of any hypothesis regarding the turbulence but follows from the fundamental equation of motion (Reynolds). Finally, for these simple cases the equation of motion permits the experimental computation of the shear stress at the wall by measuring the drop in static pressure along the flow, as may be done with very great accuracy. The latter circumstance very greatly simplifies the experimental confirmation of the theoretical resistance laws.

The careful experimental investigation of the above two cases of turbulent flow has led to a completely satisfactory application for practical purposes of the semiempirical turbulence theories (Prandtl, Kármán, Mattioli). The empirical nature of these theories lies, as is known, in the presence of the experimental constants χ (the absolute turbulence constant) and α (the nondimensional thickness of the laminar sublayer at the wall) obtained in evaluating the test data on the relation between the friction at the wall and the Reynolds number from the formulas obtained on the basis of these theories.

At the present time it may be confidently asserted that for the above-mentioned simplest cases of flow the constancy and absoluteness of these constants are facts that have been repeatedly verified. However, it still remains very uncertain to what extent the absoluteness

of these constants is maintained in passing to other cases of motion, in particular to motion, not with a negative static pressure gradient (as in the pipe and channel), but with a positive gradient – that is, to those cases characterized by the existence of inertia forces. As is known, in these cases both the velocity and the frictional stress profiles undergo radical changes. Such a case is that, for example, of the turbulent motion in the boundary layer at the upper surface of a wing. Notwithstanding the undoubted urgency of the solution of the problem of the wing boundary layer this type of motion appears very unsuitable for the purposes of generalizing the existing semiempirical turbulence theories. Thus, the frictional stress along the wing contour varies according to a law which is connected with the velocity distribution law by a very complicated integro-differential condition (Kármán).

In addition to the fact that this condition on evaluating the test data requires the carrying out of graphical differentiation of the experimental curves, a procedure which introduces an element of arbitrariness in the case of the flow about a wing, there, strictly speaking, does not exist a determinate problem since the static pressure distribution over the wing profile must in all cases be obtained from experiment. Thus the "turbulent" character of the phenomenon in the boundary layer is very much complicated by accessory circumstances of the external problem. For this reason it was considered desirable to obtain a type of flow which, while possessing all the properties associated with a variable positive pressure gradient, was most free from various external complicating circumstances.

Such a type of motion, that is, a somewhat exaggerated model of the boundary layer of a wing, may be represented by the steady flow in a straight-walled or conical diffuser. Since in this case the entire region within the diffuser is filled with the "boundary layer," the chief difficulty of the external problem – namely, the incompleteness of the equations of motion as regards the external conditions -- drops out. The equations obtained are determinate both for the velocity and the pressure distributions. Moreover, by introducing a certain assumption, for small divergence angles of the diffuser there is the possibility of greatly simplifying the investigation of the character of the change in the values of the velocities, friction, and pressures along the flow by entirely avoiding the operations associated with graphical differentiation. By investigating the possibility of generalizing the semiempirical theory to the case of the diffuser it is possible then to proceed on a surer basis to the boundary layer study. In selecting the conical diffuser for the present investigation the following facts were considered because of convenience in testing:

1. To obtain steady turbulence in the diffuser, it is necessary to have an initial inlet length ahead of it. It was convenient for this purpose to use the cylindrical pipe, already investigated in a previous paper (reference 1), which may be considered as a "diffuser of zero divergence."

2. In investigating a straight-walled diffuser with an initial inlet length, it is necessary to make the distance between its lateral parallel walls as large as possible in order to avoid the effect of increasing boundary layer on the flow in the axial plane. A large distance between the lateral walls would give such a large diffuser cross section, however, that the air intake apparatus at disposal would be unable to produce a flow with sufficiently large Reynolds number.

The present paper is divided into three parts. In the first part an attempt is made to give a theoretical analysis of the turbulent flow in a conical diffuser with small divergence angle. In the second part the procedure is described, the results of the tests conducted on conical diffusers with cone angles of 1° and 2° are presented, and an exhaustive comparison of the experimental results with the theory presented in the first part is given. In the third part an approximate method is given for estimating the deviation of the true motion from the radial pattern assumed in the first part.

I. THEORETICAL ANALYSIS OF THE TURBULENT FLOW IN A CONICAL DIFFUSER

1. Fundamental Equations of Motion

It will be assumed that a fully turbulent flow enters from a straight cylindrical pipe of radius R into a conical diffuser with angle ϑ_0 between the axis and the generator (fig. 1). To investigate this case, it is convenient to assume a spherical system of coordinates with the pole at the vertex of the cone O and with the polar axis directed along the axis of the cone. Thus the coordinates of any point M within the diffuser will be the distance from the pole r and two angles, namely, ϑ between the straight line connecting the point with the pole and the polar axis, and φ between the perpendicular dropped from the point M on the polar axis and any fixed plane containing the polar axis.

The hydrodynamic equations (Euler) in the chosen coordinates have, as is known, for the steady motion the following form:

$$\left.\begin{aligned}
v_r \frac{\partial v_r}{\partial r} + \frac{v_\vartheta}{r}\frac{\partial v_r}{\partial \vartheta} + \frac{v_\varphi}{r \sin \vartheta}\frac{\partial v_r}{r\partial \varphi} - \frac{{v_\vartheta}^2}{r} - \frac{{v_\varphi}^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\
v_r \frac{\partial v_\vartheta}{\partial r} + \frac{v_\vartheta}{r}\frac{\partial v_\vartheta}{\partial \vartheta} + \frac{v_\varphi}{r \sin \vartheta}\frac{\partial v_\vartheta}{\partial \varphi} + \frac{v_r v_\vartheta}{r} - \frac{{v_\varphi}^2}{r}\cot\vartheta &= -\frac{1}{\rho r}\frac{\partial p}{\partial \vartheta} \\
v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_\vartheta}{r}\frac{\partial v_\varphi}{\partial \vartheta} + \frac{v_\varphi}{r \sin \vartheta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r v_\varphi}{r} + \frac{v_\vartheta v_\varphi}{r}\cot\vartheta &= -\frac{1}{\rho r \sin\vartheta}\frac{\partial p}{\partial \varphi}
\end{aligned}\right\} \quad (1)$$

and the equation of continuity

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta} + \frac{1}{r \sin\vartheta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{2v_r}{r} + \frac{v_\vartheta}{r}\cot\vartheta = 0 \quad (2)$$

where v_r is the component of the velocity directed along the straight line passing through the pole and v_ϑ and v_φ are the velocity components directed along the tangents to the arcs measuring the corresponding angles; p is the static pressure in the flow, and ρ the density of the fluid. From the preceding equations is set up the equation of Reynolds for which purpose, as usual, the velocity and pressure are broken down into a mean value with respect to time and a fluctuation about the mean:

$$\left.\begin{aligned}
v_r &= \bar{v}_r + v_r' \\
v_\varphi &= \bar{v}_\varphi + v_\varphi' \\
v_\vartheta &= \bar{v}_\vartheta + v_\vartheta' \\
p &= \bar{p} + p'
\end{aligned}\right\} \quad (3)$$

Equations (1) with the aid of the continuity equation may be transformed into the following;

$$\frac{\partial(v_r^2)}{\partial r} + \frac{1}{r}\frac{\partial(v_r v_\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial(v_r v_\varphi)}{\partial\varphi} - \frac{v_\vartheta^2}{r} - \frac{v_\varphi^2}{r}$$

$$+ \frac{v_r v_\vartheta}{r}\cot\vartheta = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\partial(v_r v_\vartheta)}{\partial r} + \frac{1}{r}\frac{\partial(v_\vartheta^2)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial(v_\vartheta v_\varphi)}{\partial\varphi} + \frac{3v_r v_\vartheta}{r} + \frac{v_\vartheta^2}{r}\cot\vartheta$$

$$-\frac{v_\varphi^2}{r}\cot\vartheta = -\frac{1}{\rho r}\frac{\partial p}{\partial\vartheta}$$

$$\frac{\partial(v_\varphi v_r)}{\partial r} + \frac{1}{r}\frac{\partial(v_\varphi v_\vartheta)}{\partial\vartheta^2} + \frac{1}{r\sin\vartheta}\frac{\partial(v_\varphi)^2}{\partial\varphi} + \frac{3v_r v_\varphi}{r} + \frac{2v_\vartheta v_\varphi}{r}\cot\vartheta$$

$$= -\frac{1}{\rho r\sin\vartheta}\frac{\partial p}{\partial\varphi}$$

By substituting (3), averaging with respect to time and remembering that according to the averaging laws:

$$\overline{v_r v_\vartheta} = \bar{v}_r\bar{v}_\vartheta + \overline{v_r' v_\vartheta'}$$

$$\bar{v}' = 0, \text{ etc.}$$

the three equations of motion are obtained:

$$\frac{\partial(\overline{v}_r^{\,2})}{\partial r} + \frac{1}{r}\frac{\partial(\overline{v}_r\,\overline{v}_\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial(\overline{v}_r\,\overline{v}_\varphi)}{\partial\vartheta} - \frac{\overline{v}_\varphi^{\,2}}{r} - \frac{\overline{v}_\vartheta^{\,2}}{r} + \frac{2\overline{v}_r^{\,2}}{r}$$

$$+ \frac{\overline{v}_r\,\overline{v}_\vartheta}{r}\cot\vartheta = -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial r} + \frac{1}{\rho}\left[\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\vartheta}}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\right.$$

$$\left.\times\frac{\partial\tau_{r\varphi}}{\partial\varphi} - \frac{\tau_{\vartheta\vartheta}}{r} - \frac{\tau_{\varphi\varphi}}{r} + \frac{2\tau_{rr}}{r} + \frac{\tau_{r\vartheta}}{r}\cot\vartheta\right]$$

$$\frac{\partial(\overline{v}_r\,\overline{v}_\vartheta)}{\partial r} + \frac{1}{r}\frac{\partial\overline{v}_\vartheta^{\,2}}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial(\overline{v}_\vartheta\,\overline{v}_\varphi)}{\partial\varphi} + \frac{3\overline{v}_r\,\overline{v}_\vartheta}{r} - \frac{\overline{v}_\varphi^{\,2}}{r}\cot\vartheta$$

$$+ \frac{\overline{v}_\vartheta^{\,2}}{r}\cot\vartheta = -\frac{1}{\rho r}\frac{\partial\overline{p}}{\partial\vartheta} + \frac{1}{\rho}\left[\frac{\partial\tau_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\vartheta\vartheta}}{\partial\tau} + \frac{1}{r\sin\vartheta}\right.$$

$$\left.\times\frac{\partial\tau_{\varphi\varphi}}{\partial\varphi} + \frac{3\tau_{r\vartheta}}{r} - \left(\frac{\tau_{\varphi\varphi}}{r} - \frac{\tau_{\vartheta\vartheta}}{r}\right)\cot\vartheta\right]$$

$$\frac{\partial(\overline{v}_r\,\overline{v}_\varphi)}{\partial r} + \frac{1}{r}\frac{\partial(\overline{v}_\vartheta\,\overline{v}_\varphi)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial(\overline{v}_\varphi^{\,2})}{\partial\varphi} + \frac{3\overline{v}_r\,\overline{v}_\varphi}{r} + \frac{2\overline{v}_\vartheta\,\overline{v}_\varphi}{r}\cot\vartheta$$

$$= -\frac{1}{\rho r\sin\vartheta}\frac{\partial\overline{p}}{\partial\varphi} + \frac{1}{\rho}\left[\frac{\partial\tau_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\varphi\vartheta}}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial\tau_{\varphi\varphi}}{\partial\varphi}\right.$$

$$\left. + \frac{3\tau_{r\varphi}}{r} + \frac{2\tau_{\varphi\vartheta}}{r}\cot\vartheta\right]$$

and the continuity equation

$$\frac{\partial\overline{v}_r}{\partial r} + \frac{1}{r}\frac{\partial\overline{v}_\vartheta}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial\overline{v}_\varphi}{\partial\varphi} + \frac{2\overline{v}_r}{r} + \frac{\overline{v}_\vartheta}{r}\cot\vartheta = 0$$

where

$$\left.\begin{aligned}\tau_{r\vartheta} &= -\rho\,\overline{v_r'v_\vartheta'}\\ \tau_{r\varphi} &= -\rho\,\overline{v_r'v_\varphi'}\\ \tau_{\varphi\vartheta} &= -\rho\,\overline{v_\varphi'v_\vartheta'}\end{aligned}\right\}$$

are the tangential and

$$\left.\begin{aligned}\tau_{rr} &= -\rho\,\overline{v_r'^2}\\ \tau_{\vartheta\vartheta} &= -\rho\,\overline{v_\vartheta'^2}\\ \tau_{\varphi\varphi} &= -\rho\,\overline{v_\varphi'^2}\end{aligned}\right\}$$

the normal components of the stress tensor.

Since the Reynolds equations were derived from the Euler equations, that is, equations that do not take the viscosity into account, it is safe to assume that the tangential components of the stress tensor

$$\tau_{r\vartheta},\quad \tau_{r\varphi},\quad \text{and}\quad \tau_{\varphi\vartheta}$$

represent the sum of the viscous and turbulence parts of the stresses, because the viscous stresses, by analogy with the turbulence stresses, are also a consequence of the averaging with respect to time of the actual molecular motion.

If the analogous components of the viscous stresses are substituted (expressed in spherical coordinates in terms of the velocity gradients) in the obtained equations in place of the tangential components of the turbulence stresses, and if isotropy of the normal viscous stresses is assumed, the well-known equations of Navier-Stokes are obtained.

As is known, the system of Reynolds equations is not determinate since, for determining the ten unknown functions,

$$\overline{v}_r,\quad \overline{v}_\vartheta,\quad \overline{v}_\varphi,\quad \overline{p},\quad \tau_{r\vartheta},\quad \tau_{r\varphi},\quad \tau_{\varphi\vartheta},\quad \tau_{rr},\quad \tau_{\vartheta\vartheta},\quad \tau_{\varphi\varphi}$$

only four equations are available. Some simplifying assumptions now will be introduced. It will be safe to assume that the average motion in the diffuser is one with axial symmetry; that is, the velocity component $\overline{v}_\varphi$ and all derivatives with respect to φ are equal to zero. This assumption immediately removes the third equation from consideration and simplifies the others. Then, introduce the less obvious assumption that the mean motion is along the straight lines passing through the pole. This, of course, is a very strong assumption, approximately satisfied evidently only for small divergence angles of the diffuser, and, no doubt, requiring experimental confirmation (which will be presented later). By this assumption it may be considered that $v_\vartheta = 0$. In this case it follows that:

$$2\overline{v}_r \frac{\partial \overline{v}_r}{\partial r} + \frac{2\overline{v}_r^{\,2}}{r} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial r}$$

$$+\frac{1}{\rho}\left[\frac{1}{r}\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\vartheta}}{\partial \vartheta} - \frac{\tau_{\vartheta\vartheta}}{r} - \frac{\tau_{\varphi\varphi}}{r} + \frac{2\tau_{rr}}{r} + \frac{\tau_{r\vartheta}}{r}\cot\vartheta\right]$$

$$0 = -\frac{1}{\rho r}\frac{\partial \overline{p}}{\partial \vartheta} + \frac{1}{\rho}\left[\frac{\partial \tau_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\vartheta\vartheta}}{\partial \vartheta} + \frac{3\tau_{r\vartheta}}{r} - \left(\frac{\tau_{\varphi\varphi}}{r} - \frac{\tau_{\vartheta\vartheta}}{r}\right)\cot\vartheta\right]$$

Introduce the assumption of isotropy of the normal components of the turbulent stresses, that is, that the equations are approximately satisfied

$$\tau_{rr} \approx \tau_{\vartheta\vartheta} \approx \tau_{\varphi\varphi}$$

Then there is obtained from the first equation, letting for simplicity $v_r = u$ and $\tau_{r\vartheta} = \tau$

$$2u\frac{\partial u}{\partial r} + \frac{2u^2}{r} = -\frac{1}{\rho}\frac{\partial(\overline{p} - \tau_{rr})}{\partial r} + \frac{1}{\rho r \sin\vartheta}\frac{\partial(\tau \sin\vartheta)}{\partial \vartheta}$$

where $\overline{p} - \tau_{rr}$ represents some total pressure which at the diffuser wall, where the fluctuations vanish, is equal to the static. Setting $\overline{p} - \tau_{rr} = p_1$, and assuming in view of the smallness of the angles $\sin \vartheta \approx \vartheta$ gives:

$$2u \frac{\partial u}{\partial r} + \frac{2u^2}{r} = -\frac{1}{\rho} \frac{\partial p_1}{\partial r} + \frac{1}{\rho r \vartheta} \frac{\partial(\tau \vartheta)}{\partial \vartheta} \tag{4}$$

The second equation containing the gradient of $\overline{p}$ transverse to the diffuser is excluded entirely from consideration. This equation is obtained by taking the sum of the projections of all forces acting on the fluid in the direction perpendicular to the mean radial motion. Therefore in introducing the assumption of the radial character of the motion, this equation suffers to a considerably greater degree than the first, obtained from considering the projections of all the forces in the direction of the initial motion. It must be supposed that the neglected terms in the second equation, due to the assumption of radial motion, which contain the component v_ϑ are of the same order of magnitude as the remaining terms containing $\frac{\partial \overline{p}}{\partial \vartheta}$ and the difference $\tau_{\varphi\varphi} - \tau_{\vartheta\vartheta}$. Thus the retention of the second equation must be considered as unsuitable and even harmful. In what follows it will be seen that the remaining equations are entirely sufficient for solving the problem for the assumptions made.

The continuity equation under the assumption of axial symmetry and radial character of the mean motion becomes:

$$\frac{\partial u}{\partial r} + \frac{2u}{r} = 0$$

This equation may be easily integrated with respect to r and its general integral will be:

$$u = \frac{t(\vartheta)}{r^2} \tag{5}$$

where t is an arbitrary function of ϑ only. Expression (5) shows

that under these assumptions similar velocity profiles should be obtained over the entire length of the diffuser and that along each radial line of flow the velocity varies in the same manner as for the motion of an ideal fluid - that is, inversely proportional to the square of the distance from the source.

Substituting equation (5) in the initial equation (4) results in the following equation after combining similar terms and multiplying by r^5:

$$2t^2 = \frac{1}{\rho}\frac{\partial p_1}{\partial r} r^5 - \frac{r^4}{\rho\vartheta}\frac{\partial(\tau\vartheta)}{\partial\vartheta}$$

From the equation obtained two very important conclusions can be drawn on the change in pressure and intensity of friction along the diffuser. Since the left side of the equation does not depend on r, it must be assumed that the right side likewise is independent of r. This can be the case only if $\partial p_1/\partial r$ is inversely proportional to r^5 and τ is inversely proportional to r^4.

By setting

$$G = \frac{1}{\rho}\frac{\partial p_1}{\partial r} r^5 \quad \text{and} \quad f = \tau r^4 \tag{6}$$

where G and f according to what was said above do not depend on r, the above equation of motion may be written as the usual one:

$$2t^2 = G - \frac{1}{\rho\vartheta}\frac{d(f\vartheta)}{d\vartheta} \tag{7}$$

The first expression in (6) may easily be integrated with respect to r. There is obtained

$$p_1 - p_{1_0} = -\frac{1}{4}\rho G\left(\frac{1}{r^4} - \frac{1}{r_0^{\,4}}\right) \tag{8}$$

where p_{1_0} is the absolute value of the generalized pressure at any point of the flow having the coordinate r_0. The expression (8) giving the law of variation of this pressure along the diffuser, together with the law of drop in velocity (5), is very suitable for the experimental checking of the fundamental assumption of radial flow. Use will be made of this expression later.

It was assumed that the pressure p_1 is the difference between the static pressure and the normal turbulent stress:

$$p_1 = \overline{p} - \tau_{rr}$$

Thus the magnitude G, strictly speaking, is determined as

$$G = \frac{r^5}{\rho}\left(\frac{\partial \overline{p}}{\partial r} - \frac{\partial \tau_{rr}}{\partial r}\right)$$

It is assumed that the gradient along r of τ_{rr} is small by comparison with the gradient of the pressure p. This makes it possible to consider

$$G \approx \frac{r^5}{\rho} \frac{\partial \overline{p}}{\partial r}$$

In studying the boundary layer the assumption is usually made that the static pressure does not vary over the thickness of the boundary layer. This is excellently confirmed by experiment. Applying an analogous assumption to this case, let

$$\frac{\partial \overline{p}}{\partial \vartheta} = 0$$

This at once leads to the result that G = constant. Multiplying equation (7) by ϑ and integrating from $\vartheta = 0$ (axis of diffuser) to a variable ϑ gives

$$2\int_0^{\vartheta} t^2 \vartheta d\vartheta = \frac{G\vartheta^2}{2} - \frac{1}{\rho} f\vartheta$$

Transform to nondimensional variables for which purpose the angle is referred to its maximum value ϑ_0, letting

$$\frac{\vartheta}{\vartheta_0} = \xi$$

The velocity u will be referred to the "dynamic velocity of friction" at the wall $v_* = \sqrt{\tau_0/\rho}$ where τ_0 is the frictional intensity at the wall, and then $u/v_* = \varphi$. Since $u = t/r^2$ and

$$v_* = \sqrt{\tau_0/\rho} = \sqrt{f_0/r^4\rho} = \frac{1}{r^2}\sqrt{f_0/\rho}$$

$t = b\varphi$ is obtained where $b = \sqrt{f_0/\rho}$. Substituting in the last expression yields

$$2\int_0^\xi \varphi^2 \xi \, d\xi = \frac{G\xi^2}{2b^2} - \frac{1}{\vartheta_0}\frac{f\xi}{\rho b^2} \tag{9}$$

On substituting in the second term of the right side $b = \sqrt{f_0/\rho}$, there is obtained the generalized law of the distribution of the frictional stress transverse to the diffuser:

$$\frac{f}{f_0} = \frac{\tau}{\tau_0} = \frac{1}{\xi}\left[2\vartheta_0\int^\xi \varphi^2 \xi \, d\xi - \frac{\vartheta_0 G\xi^2}{2b^2}\right] \tag{10}$$

Consider this expression. For $\xi = 0$ on the diffuser axis there obtains, on applying the rule of L'Hospital,

$$\frac{f}{f_0} = \lim_{\xi \to 0} \frac{2\vartheta_0 \int_0^\xi \varphi^2 \xi \, d\xi}{\xi} = \lim_{\xi \to 0} \frac{\varphi^2 \xi \times 2\vartheta_0}{1} = 0$$

At the diffuser wall for $\xi = 1$, it is seen that $f = f_o$ and

$$2\vartheta_o \int_0^1 \varphi^2 \xi d\xi - \frac{\vartheta_o G}{2b^2} = 1 \tag{11}$$

which gives the relation between the integrals of the inertia forces, the frictional forces, and the forces arising from the drop in static pressure.

In order to estimate the shape of the curve of friction distribution across the diffuser for various signs of the pressure gradient, the derivative $\frac{\partial}{\partial \xi}\left(\frac{f}{f_o}\right)$ will be found for $\xi = 1$, that is, at the diffuser wall. Differentiating (10) with respect to ξ gives:

$$\frac{d}{d\xi}\left(\frac{f}{f_o}\right) = \vartheta_o \left[2\varphi^2 - \frac{G}{2b^2} - \frac{2}{\xi^2}\int_0^{\xi} \varphi^2 \xi d\xi\right] \tag{12}$$

Substituting $\xi = 1$ and bearing in mind that at the wall $\varphi = 0$ yields

$$\left[\frac{d}{d\xi}\left(\frac{f}{f_o}\right)\right]_{\xi=1} = -\left(\frac{\vartheta_o G}{b^2} + 1\right)$$

For not too small divergence angles of the diffuser, the sign of the magnitude in parenthesis usually is entirely determined by the first term. Thus, the sign of the derivative of the curve of friction distribution will be negative for $G > 0$; that is, for positive pressure gradient and the curve of friction distribution $f/f_o = f(\xi)$ will have the form shown on figure 2. As ϑ_o approaches zero, that is, in passing from the diffuser to the straight pipe, there is obtained from (10) bearing in mind (11):

$$\frac{\tau}{\tau_o} = \xi$$

that is, a linear distribution of the frictional stress. This straight line, corresponding to the negative pressure gradient, is also shown in figure 2.

It is natural to assume that the method explained is suitable also for considering the motion in a converging pipe where $\vartheta_0 < 0$. In this case the point of intersection of the radial lines of flow lies on the other side of the section considered. Since r in this case will be reckoned opposite the flow direction, it follows that

$$\frac{\partial \bar{p}}{\partial r} > 0$$

although the pressure drops along the flow. Thus for the converging pipe on the basis of (12), $\frac{d}{d\xi}\left(\frac{f}{f_0}\right)$ will be positive for $\xi = 1$ (fig. 2).

It is not difficult to conjecture, on the basis of the above discussion, that a diffuser may be imagined for which there is no change in pressure along the flow; that is, $G = 0$. The divergence angle of such a diffuser will be determined on the basis of (11) as[1]

$$\vartheta_{0(p=\text{const})} = \frac{1}{2\int_0^1 \varphi^2 \xi \, d\xi} \tag{13}$$

K. K. Fedyaevsky, in his paper on the boundary layer of a wing (reference 2), has shown that for the external problem in the absence of a pressure gradient along the surface of the body (flat plate) the derivative of the curve of friction distribution in the direction perpendicular to the surface at the wall is equal to zero. It is interesting that for the diffuser this assumption is not found to be correct. According to expression (12) the derivative of the friction distribution at the wall in the absence of a pressure gradient along the diffuser – that is, for $G = 0$ – is equal to

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[1]Such motion represents the boundary layer of an infinitely thin flat plate set at zero angle of attack.

$$\left[\frac{d}{d\xi}\left(\frac{f}{f_o}\right)\right]_{\substack{\xi=1\\ p=\text{const}}} = -1$$

The corresponding curve is shown on figure 2.

Consider what the shape of the curve of friction distribution will be near the diffuser axis, that is, for $\xi = 0$. The first derivative of the curve of friction distribution for $\xi = 0$ is obtained from (12) on substituting $\xi = 0$. Evaluating the indeterminate expression

$$\lim_{\xi \to 0} \frac{1}{\xi^2} \int_0^\xi \varphi^2 \xi \, d\xi$$

gives:

$$\lim_{\xi \to 0} \frac{2}{\xi^2} \int_0^\xi \varphi^2 \xi \, d\xi = 2 \lim_{\xi \to 0} \frac{\varphi^2 \xi}{2\xi} = \varphi^2{}_m$$

On substituting in (11) there is obtained

$$\left[\frac{d}{d\xi}\left(\frac{f}{f_o}\right)\right]_{\xi=0} = \vartheta_o \left[\varphi_m{}^2 - \frac{G}{2b^2}\right]$$

where φ_m is the value of the nondimensional velocity u/v_* on the diffuser axis. Thus, the value of the derivative of the curve of friction distribution on the diffuser axis and, of course, also in the general case is not equal to zero. In the boundary layer, according to the investigation of Fedyaevsky, this derivative is equal to zero. This sums up the conclusions which may be drawn from the incomplete equations of motion on the basis of the assumed hypotheses.

2. Completion of Equation of Motion

In order to obtain the profile of the velocity distribution across the diffuser and the resistance law, it is necessary to render the fundamental equation (9) determinate by connecting in some way the generalized frictional stress $f = \tau r^4$ with the remaining variables. This, as usual, is attained by introducing some semiempirical assumption with regard to the turbulence.

It is possible, of course, to take any one of the semiempirical turbulence theories available at the present time (Prandtl, Karman, Mattioli) and generalize it to the case of the friction in a diffuser. This generalization to a first approximation (evidently quite satistory for small divergence angles) may be brought about by simply passing from cylindrical coordinates to polar. At first this method was followed in an attempt to generalize the theory of Mattioli. The unusual complexity, however, in the solution of the differential equation obtained, the impossibility of obtaining the velocity profile in a finite form suitable for practical application,[1] and so forth, made it necessary to drop this method and seek another.

Still another consideration led to this resolve. All empirical theories without exception are built on a very shaky physical basis. Neither the assumption of Karman on the similarity of the fields of velocity fluctuations nor the assumption of Mattioli on the transport of momentum can at the present time be supported by any but those authors. It is very significant that at the present time papers devoted to the analysis and improvement of these theories no longer appear. All this indicates that interest in these theories has dropped sharply and that they are now in the passing stage. On the basis of what has been said it may be asked whether it is worth the effort, by overcoming the huge computational difficulties, to generalize any of the theories mentioned to the case of flow in a diffuser. Even after having overcome all the computational difficulties and having obtained excellent agreement of the results of computation with experiment, no progress in learning the mechanism of the turbulent motion will have been made, for it would be very difficult to establish on what grounds the good results were obtained, whether on account of the correctness of the generalized semiempirical theory or on account of the fact that the particular nature of the flow in the diffuser was already sufficiently taken into account by the incomplete equation of motion. It would, however, be a mistake to think that the semiempirical theories of

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[1]As is known, even for the case of flow in a pipe, the theory of Mattioli leads to incomputable analytical quadratures.

turbulence which have been mentioned have contributed absolutely nothing to the understanding of the physical picture of the phenomenon. It is sufficient to recall that all these theories lead to the same resistance law for the straight pipe:

$$\frac{1}{\sqrt{C_f}} = \frac{1}{\chi\sqrt{2}} \log(R_m \sqrt{C_f}) + A \tag{14}$$

(where $C_f = \dfrac{\tau_o}{\rho \dfrac{u^2{}_m}{2}}$, $R_m = \dfrac{Ru_m}{\nu}$, R is the radius of the pipe, χ and A are universal constants), which is excellently confirmed by numerous tests. This leads to the supposition that all these theories contain some common element that correctly represents the phenomenon. In order to explain what constitutes this common element, it is recalled that Prandtl connected the frictional shear with the derivative of the mean velocity by means of the relation

$$\tau = -\rho\,\overline{u'v'} = -\rho l^2 \left(\frac{du}{dy}\right)^2 \tag{15}$$

where l, a linear magnitude denoted by Prandtl as the "mixing length" (Mischungsweg), is a function of the distance y from the wall of the pipe or channel. From its meaning l should become zero at the wall where no mixing can take place. Prandtl made the very simple and elegant assumption that follows from the possibility of developing the function $l = l(y)$ into a series at the wall; namely, he assumed that to a first approximation

$$l = \chi y \tag{16}$$

where χ, a universal turbulence constant, is the first derivative of the mixing length with respect to y at the wall, that is, for $y = 0$

$$\chi = \left(\frac{dl}{dy}\right)_{y=0} \tag{17}$$

As is known, the assumption (15) led Prandtl to the "universal velocity profile" and to the resistance law (14). With the aid of the latter the magnitude of χ was found and its absolute character established.

Since from other theories (Kármán, Mattioli) the same formula was obtained, it is to be expected that for these theories condition (17) is likewise satisfied - which statement will be proved.

As is known, Kármán obtains a relation connecting the mixing length with the derivatives of the velocity in the form:

$$l = \chi \frac{u'}{u''} = \chi R \frac{\varphi'}{\varphi''}$$

and the velocity distribution in the form:

$$\varphi_m - \varphi = -\frac{1}{\chi}\left[\log(1 - \sqrt{\xi}) + \sqrt{\xi}\right]$$

By finding the derivatives φ' and φ'' and substituting in the expression for l there is obtained

$$l = 2\chi R(\sqrt{\xi} - \xi)$$

Differentiating with respect to ξ gives

$$\frac{dl}{d\xi} = 2\chi R\left(\frac{1}{2\sqrt{\xi}} - 1\right)$$

By substituting $\xi = 1$ and remembering that $d\xi = d(1 - y/R) = - dy/R$ there is obtained

$$\left(\frac{dl}{dy}\right)_{y=o} = \chi$$

as was to be proved.

To verify expression (17) according to the theory of Mattioli, equation (15) is solved for l and differentiated with respect to y, where

$$\tau = \tau_0 \left(1 - \frac{y}{R} \right)$$

$$\frac{dl}{dy} = -v_* \left[\frac{1}{2R \sqrt{1 - \frac{y}{R}} \frac{du}{dy}} - \frac{d}{dy} \left(\frac{1}{\frac{du}{dy}} \right) \sqrt{1 - \frac{y}{R}} \right]$$

Introducing the variables φ and ξ yields:

$$\frac{dl}{dy} = - \left[\frac{1}{\varphi' \sqrt{\xi}} - \frac{d}{d\xi} \left(\frac{1}{\varphi'} \right) \sqrt{\xi} \right]$$

As is known, the theory of Mattioli for large Reynolds numbers leads to the following relations:

(a) $\varphi' = ce^{\chi\Theta}$, where $\Theta = \dfrac{\varphi_m - \varphi - \beta}{\xi}$ $\qquad \beta = \text{constant}$, $c = \text{constant}$

(b) $$\frac{1}{\xi} \frac{d\xi}{d\Theta} = - \frac{1}{\varphi' + \Theta}$$

Making use of these relations gives, from the expression for dl/dy,

$$\frac{dl}{dy} = \frac{1}{\sqrt{\xi}} \left[\chi \left(1 + \frac{\Theta}{ce^{\chi\Theta}} \right) - \frac{1}{ce^{\chi\Theta}} \right]$$

Since for the case where the viscosity is neglected (large Reynolds numbers), the velocity at the wall approaches ∞,

$$\lim_{\xi \to 0} \Theta = \infty$$

$$\lim_{\Theta \to \infty} \frac{\Theta}{ce^{X\Theta}} = 0$$

$$\left(\frac{dl}{dy} \right)_{\xi=1} = X$$

as was to be proved.

It is thus evident that the fundamental element common to the foregoing theories is the fact that the first derivative of the curve of the mixing length against the distance from the wall has at the wall, or more accurately, at the edge of the laminar layer, a constant absolute value X independent of any variables.

Since the resistance formula, which is a consequence of this assumption, is in excellent agreement with the experiment, it may be considered that also for the diffuser and, in general, for any turbulent flow near the wall

$$\left(\frac{dl}{dy} \right)_{y=0} = X = \text{constant}$$

Moreover, numerous experimental investigators (Dönch, Nikuradse, Frietsche) have established two further significant facts: (1) the curve $\frac{l}{R} = f\left(\frac{y}{R} \right)$ is found to be almost absolute near the wall for the most varied cases of flow and (2) on the axis of the channel or pipe or on the edge of the boundary layer $\frac{l}{R}$ = constant ≈ 0.14.

On the basis of all that has been said, it is not necessary in the present paper to make use of any developed hypothesis of turbulence except the formula of Prandtl (15) but a method will be used that permits obtaining the simplest and clearest results.

Up to now it has been understood that τ represents the total frictional stress:

$$\tau = \tau_{lam} + \tau_{turb}$$

As earlier investigations (references 1, 3, and 4) have shown, it may be considered that for practical Reynolds numbers the effect of the viscous friction is negligible in comparison with the turbulent friction. For this reason, in what follows, by τ will be understood only its turbulent part.

The formula of Prandtl will be reduced to a form independent of r. On assuming, according to the foregoing discussion, the ratio l/R (where R is the local radius of the diffuser cross section, $R = r\vartheta_0$) independent of r, to be a function of ϑ, from equation (15) there is obtained

$$\tau = -\rho r^2 \vartheta_0{}^2 \left(\frac{l}{R}\right)^2 \frac{1}{r^6 \vartheta_0{}^2} \left(\frac{dt}{d\xi}\right)^2 = -\rho \frac{1}{r^4} \left(\frac{l}{R}\right)^2 \left(\frac{dt}{d\xi}\right)^2$$

It is seen that on the basis of the assumption of the absoluteness of the curve $\frac{l}{R} = f(\xi)$ it was found from the formula of Prandtl that the turbulent part of the frictional stress, in the same manner as the total frictional stress, should be inversely proportional to r^4.

The formula of Prandtl is reduced to nondimensional form by letting $t = b\varphi$:

$$\frac{f}{\rho b^2} = -\left(\frac{l}{R}\right)^2 \varphi'^2$$

where $\varphi' = \frac{d\varphi}{d\xi}$. Substituting the expression obtained in the fundamental equation of motion in the form of equation (9) gives

$$2 \int_0^\xi \varphi^2 \xi \, d\xi = \frac{G\xi^2}{2b^2} + \frac{1}{\vartheta_0} \left(\frac{l}{R}\right)^2 \xi \varphi'^2 \qquad (18)$$

It will be necessary to obtain $\left(\frac{l}{R}\right) = f(\xi)$ in the form of a power function

$$\frac{l}{R} = a\xi^n + m \tag{19}$$

where the unknown constants a, m, and n are obtained from the foregoing considerations:

(a) for $\xi = 1$ (at the wall) $\frac{l}{R} = 0$

(b) for $\xi = 1$ (at the wall) $\frac{dl}{dy} = -\frac{d}{d\xi}\left(\frac{l}{R}\right) = X$

(c) for $\xi = 0$ (at the axis) $\frac{l}{R} = 0.14$

By making use of the first condition there is obtained

$$a + m = 0 \tag{I}$$

Differentiating (19) and substituting $\xi = 1$ according to the second condition gives

$$an = -X \tag{II}$$

According to the third condition,

$$m = 0.14 \tag{III}$$

Therefore, $a = -0.14$ and $n = -\frac{X}{a} = \frac{X}{0.14}$. Since the value of X is usually of the order of 0.43 to 0.44 (according to the resistance law derived for the pipe), the approximate result is:

$$n \approx 3$$

and the required function $\frac{l}{R} = f(\xi)$ will be a cubic parabola

$$\frac{l}{R} = \frac{\chi}{3} (1 - \xi^3)$$

By substituting the obtained relation in the fundamental equation (18) there is obtained a differential equation the solution of which should give $\varphi = \varphi(\xi)$, that is, the curve of velocity distribution:

$$\frac{2}{\xi^2} \int_0^\xi \varphi^2 \xi \, d\xi = \frac{G}{2b^2} + \frac{\chi^2}{9\vartheta_0} \frac{(1 - \xi^3)^2}{\xi} \varphi'^2 \tag{21}$$

3. Velocity Distribution

Integration of the obtained equation in finite form is not possible. For this reason it is necessary to proceed to an approximate integration. By integrating the inertia integral on the left in the form of a power relation and setting

$$\frac{1}{\xi^2} \int_0^\xi \varphi^2 \xi \, d\xi = J$$

it is assumed that

$$J = a_1 + m_1 \xi^k \tag{22}$$

The constants a_1 and m_1 will be determined from the boundary conditions for J. For $\xi = 0$,

$$\lim_{\xi \to 0} J = \lim_{\xi \to 0} \frac{\varphi^2 \xi}{2\xi} = \frac{\varphi^2_m}{2}$$

for $\xi = 1$

$$J = \int_0^1 \varphi^2 \xi d\xi = \text{constant} = J_0$$

Then

$$\left.\begin{aligned} a_1 &= \frac{\varphi_m^2}{2} \\ a_1 + m_1 &= J_0 \end{aligned}\right\} m_1 = J_0 - \frac{\varphi_m^2}{2}$$

and

$$J = \frac{\varphi_m^2}{2} + \left(J_0 - \frac{\varphi_m^2}{2} \right) \xi^k$$

It is practically more convenient in the preceding expressions to pass from the universal velocities φ and φ_m to the nondimensional

$$\frac{u}{u_m} = \frac{\varphi}{\varphi_m}$$

which are directly measured in the tests. On dividing the above expression by φ_m^2 and letting

$$I = \frac{J}{\varphi_m^2} = \frac{1}{\xi^2} \int_0^\xi \left(\frac{u}{u_m} \right)^2 \xi d\xi$$

there is obtained

$$I = \frac{1}{2} + \left(I_0 - \frac{1}{2} \right) \xi^k$$

Substituting in equation (21) the expression for I and solving for the first derivative of the nondimensional velocity yields

$$\frac{d\varphi}{d\xi} = -\frac{3}{\chi}\sqrt{2\varphi_m{}^2\vartheta_o\left[\frac{1}{2} + \left(I_o - \frac{1}{2}\right)\xi^k\right] - \frac{G\vartheta_o}{2b^2}}\ \frac{\sqrt{\xi}}{1 - \xi^3} \tag{23}$$[1]

According to expression (11)

$$\frac{G\vartheta_o}{2b^2} = 2\vartheta_o\varphi_m{}^2 I_o - 1$$

Substituting this expression in equation (23) yields

$$\frac{d\varphi}{d\xi} = -\frac{3}{\chi}\sqrt{\vartheta_o\ \varphi_m{}^2\ (1 - 2I_o)(1 - \xi^k) + 1}\ \frac{\sqrt{\xi}}{1 - \xi^3} \tag{24}$$

The expression $\vartheta_o\varphi_m{}^2(1 - 2I_o)$ completely characterizes the flow state in the diffuser, since it takes account both of the geometric (ϑ_o) and the dynamic parameters φ_m and I_o depending on the Reynolds number.

Let

$$\vartheta_o\varphi_m{}^2(1 - 2I_o) = D$$

and denote this nondimensional magnitude as the "diffuser parameter." It is not difficult to see that for the "zero diffuser" (pipe) $D = 0$. For the case of flow of an ideal fluid in a diffuser, that is, when the velocity does not depend on ξ and at each section $u = u_m$,

$$I = \int_0^1 \xi d\xi = \frac{1}{2}$$

[1]The minus sign before the square root is chosen from the consideration that over the entire range of variation of ξ the derivative $\dfrac{d\varphi}{d\xi}$ is less than zero.

In the case of a diffuser and a converging pipe with turbulent flow where in computing I_o the function $\frac{u}{u_m}$ entering the integral is always less than unity, the result is

$$I_o < \frac{1}{2}$$

for which reason $D > 0$ for the diffuser and $D < 0$ for the convergent pipe $(\vartheta_o < 0)$. Integrating (24) with respect to ξ gives

$$\varphi_m - \varphi = \frac{3}{\chi} \int_0^{\xi} \frac{\sqrt{\xi}\,\sqrt{D(1-\xi^k) + 1}}{1 - \xi^3}\, d\xi \tag{25}$$

Nothing, as yet, has been said regarding the value of the exponent k in the integration of the inertia integral. It is convenient to proceed as follows: Assuming any value of k, to compute the integral (25); then, having the relation $\frac{u}{u_m} = f(\xi)$, to set up the values $I = f(\xi)$ and choose a new value for k in better agreement with the foregoing relation. By repeating this process several times it is not difficult to arrive at a value of k which best satisfies both expression (25) and the approximation $I = f(\xi)$. By this procedure, however, it is necessary to compute the integral (25) graphically, since finite computation is possible only for a few values of k.

In attempting to obtain the velocity distribution formula in finite form, it was assumed that $k = \frac{3}{2}$ which, as will be seen later, is in good agreement with all the conditions discussed. For $k = \frac{3}{2}$ the integral (25) is obtained in finite form by the substitution

$$\sqrt{D\left(1 - \xi^{\frac{3}{2}}\right) + 1} = x$$

leading to the integral of a rational fraction. Without going into detail, this gives the final result:

$$\varphi_m - \varphi = \frac{1}{\chi}\left\{ \log \frac{(x+1)(\sqrt{D+1}-1)}{(x-1)(\sqrt{D+1}+1)} - 2\sqrt{2D+1}\left[\tanh^{-1}\frac{x}{\sqrt{2D+1}} - \tanh^{-1}\sqrt{\frac{D+1}{2D+1}}\right]\right\} \tag{26}$$

The proposed method of approximating the mixing length function by a cubical parabola gives a very simple law of velocity distribution for the straight pipe. Setting in expression (25) $D = 0$ and carrying out the integration gives for the pipe

$$\varphi_m - \varphi = \frac{2}{\chi}\tanh^{-1}\left(\xi^{\frac{3}{2}}\right) \tag{27}$$

This very simple formula, as will be shown in part II, is excellently verified by experiment.

It is of interest that the absolute character (independent of the Reynolds number) of the velocity distribution profile in the form

$$\varphi_m - \varphi = f(\xi)$$

in the case of the pipe does not hold for the diffuser, for which case the "diffuser parameter" depending on the Reynolds number enters the velocity distribution (formula (26)).

4. Resistance Law

To make use of the foregoing derived formula for the velocity distribution in the diffuser, it is necessary to be given the diffuser parameter

$$D = \vartheta_o \varphi_m^{\ 2}(1 - 2I_o)$$

that is, the values I_o and φ_m. The value of the inertia integral

$$I_0 = \int_0^1 \left(\frac{u}{u_m}\right)^2 \xi d\xi$$

will be known, since the velocity profile is known. It is suggested that an approximate value for I_0 first be assumed and then, having obtained the profile (26), to correct this value. Generally the successive approximations converge very rapidly, since the values of I_0 depend little on the shape of the velocity profile.

It remains, for computing D, to assume the value $\varphi_m = \frac{u_m}{v_*}$, which must be associated with some characteristic of the flow, for example, the Reynolds number. This means that it is necessary to find the resistance law.

To obtain the resistance law, the solution obtained must be connected with the laminar layer near the wall. Since in the laminar layer only the effect of the viscosity should be assumed, consider where these equations lead to if the viscosity friction is considered instead of the turbulent friction.

By retaining for the present the assumption of radial flow $(v = 0)$ also for the case of the laminar sublayer it is found that the intensity of the laminar friction is expressed as

$$\tau_{lam} = -\mu \frac{1}{r} \frac{du}{d\vartheta} = -\mu \frac{1}{r^3} \frac{dt}{d\vartheta} \tag{28}$$

The assumption of radial flow leads to the result that the laminar friction, in contrast to the turbulent, varies inversely proportionally to the third, and not the fourth, power of r.

The equations of the laminar motion on the assumption of radial flow, on the basis of the fundamental equations and the above expression for the friction, become

$$\frac{2t^2}{r^5} = -\frac{1}{\rho} \frac{\partial p}{\partial r} - \frac{\nu}{r^4} \left[\frac{d^2t}{d\vartheta^2} - \frac{1}{\vartheta} \frac{dt}{d\vartheta}\right]$$

$$\frac{1}{\rho} \frac{\partial p}{\partial \vartheta} = -\frac{2\nu}{r^3} \frac{dt}{d\vartheta}$$

On eliminating the pressure p by the method of cross differentiation there is obtained after dividing by $\frac{1}{r^4}$:

$$\frac{4t}{r}\frac{dt}{d\vartheta} = \nu\left[\frac{d^3t}{d\vartheta^3} - 6\frac{dt}{d\vartheta} - \frac{d}{d\vartheta}\left(\frac{1}{\vartheta}\frac{dt}{d\vartheta}\right)\right]$$

In the foregoing expression the right side does not depend on r; whereas the left side is inversely proportional to r. This absurdity clearly shows that the viscous fluid cannot flow radially in a conical diffuser. With this result, it becomes evident that radial flow in a turbulent fluid is, strictly speaking, impossible because the internal friction in a turbulent flow is the sum of the turbulent and the viscous friction. It is not difficult to see, however, that the deviation from radiality in the main body of the flow should be negligible, being of the same order of smallness as the proportion of the viscous friction is to the total friction. In other words, the flow in the diffuser may be thought of as a strictly radial, "purely turbulent" flow with superposed small nonradial disturbances arising from the viscosity.

The greatest deviation from radiality should be expected near the wall where the proportion of the viscous friction is particularly large. In order to form a very approximate picture of the flow in the laminar sublayer the following reasoning is given. It has been seen that for the radial flow assumption valid in the main body of the flow the total friction should, on the basis of the Reynolds equations, vary inversely proportionally to r^4. On the other hand, on the basis of the Prandtl formula and the assumption of universality of the nondimensional mixing length curve the purely turbulent part of the friction should likewise be inversely proportional to r^4. This justifies the assumption that the laminar part of the friction should have the same property with the same degree of accuracy. With the object of obtaining a smooth profile of the friction distribution, it is permissible to extend this property also to the purely laminar layer, that is, ascume that

$$\tau_{lam} \sim \frac{1}{r^2}$$

In this way, the assumption made in solving the problem of the turbulent flow in a straight pipe is extended. It was assumed there that at the wall there is a viscous layer the motion in which is subject to the equations of a viscous fluid, but the value of the friction at the wall, entering as a boundary condition, is determined by the turbulence law (resistance formula). Here, by extending the foregoing assumption,

it is considered that not only the value of the friction at the wall but also the law of the distribution of the friction along the diffuser is determined by the main turbulent part of the flow.

In the laminar layer, the intensity of the frictional stress in terms of the derivatives of the velocity in spherical coordinates is expressed:

$$\tau_{lam} = \mu\left(\frac{1}{r}\ \frac{\partial u}{\partial \vartheta} + \frac{1}{r}\ \frac{\partial(vr)}{\partial r}\right)$$

On substituting, on the basis of the above assumption, $\tau_{lam} = \frac{f_{lam}}{r^4}$, it is found that for the obtained expression to be independent of r, it is necessary that in the laminar layer the velocity components u and v be inversely proportional to r^3, a condition which also does not contradict the continuity equation. Thus, near the wall there should occur a somewhat more rapid decrease in velocity along the diffuser than in the regions with rapid flow. With the velocity near the wall decreasing more rapidly, the direction of the component v, from considerations of continuity, should be from the wall to the axis; that is, the direction of a line of flow near the wall deviates from the radial direction inward.

The angular deviation of the lines of flow from the radial directions (very small, of course, in absolute magnitude) near the wall where the effect of the viscous friction is large may be comparable with the small divergence angle of the diffuser. In this case it is to be expected that this deviation may affect the main turbulence of the flow in which the mean velocity vectors deviate somewhat from the geometric radii. The problem of the deviations from the radial direction in part III of the present paper will be considered later.

It may be well, now, to proceed with the direct derivation of the resistance formulas by the general method of considering the velocities at the boundary of the laminar layer. The velocity distribution $u = f(\vartheta)$ in the laminar layer is assumed to be linear:

$$u = \left(\frac{\partial u}{\partial y}\right)_{y=o} y$$

where y is the distance from the wall. This formula corresponds to keeping the first significant term in the development of the function

$$u = f(y)$$

in a series, a procedure which is entirely permissible in view of the extremely small thickness of the layer. Substituting

$$y = r(\vartheta_o - \vartheta) = r\vartheta_o(1 - \xi)$$

and expressing $\left(\frac{\partial u}{\partial y}\right)_{y=o}$ through the frictional shear at the wall:

$$\left(\frac{\partial u}{\partial y}\right)_{y=o} = \frac{\tau_o}{\rho\nu} = \frac{f_o}{r^4\rho\nu} = \frac{b^2}{r^4\nu}$$

gives

$$u = \frac{b^2\vartheta_o}{r^3\nu}(1 - \xi) \tag{29}$$

The thickness of the laminar layer is assumed to follow the well-known law of Kármán:

$$\delta = \alpha \frac{\nu}{v_*}$$

where, according to Kármán's assumption, α is a universal constant.[1] The value $\xi = \xi^*$ corresponding to the edge of the layer will then be

$$\xi^* = 1 - \frac{\delta}{r\vartheta_o} = 1 - \alpha\frac{\nu r}{b\vartheta_o}$$

[1]As K. K. Fediaevsky has shown, the assumption of the constancy of α corresponds to the assumption of the constancy of the critical Reynolds number computed for the thickness of the layer. There is therefore every reason to expect that in passing from the pipe to the diffuser the constant α does not appreciably change.

The nondimensional velocity $\varphi = \frac{u}{v^*}$ at the edge of the layer is obtained from equation (29)

$$\varphi_\delta = \frac{u_\delta}{v_*} = \frac{b\vartheta_o}{r\nu}\left(1 - \xi^*\right) = \alpha$$

The equations

$$\varphi = \varphi_\delta = \alpha$$

and

$$\xi = \xi^* = 1 - \alpha\frac{\nu r}{b\vartheta_o}$$

will be substituted in the formula for the velocity distribution (26). First of all, to find $x = x^*$ corresponding to $\xi = \xi^*$, apply the approximate expression for the square root as is always possible because of the extreme smallness of the magnitude $\alpha\frac{\nu r}{b\vartheta_o}$ compared with unity. There is obtained:

$$x^* = \sqrt{D\left(1 - \xi^{*\frac{3}{2}}\right) + 1} \approx 1 + \frac{D}{2}\left(1 - \xi^{*\frac{3}{2}}\right)$$

By proceeding similarly with the power of ξ^* there is obtained

$$\xi^{*\frac{3}{2}} \approx 1 - \frac{3}{2}\,\frac{\alpha\nu r}{b\vartheta_o}$$

and

$$x^* \approx 1 + \frac{3}{4}\,\frac{D\alpha\nu r}{b\vartheta_o}$$

In substituting $x = x^*$ in expression (26), the term $\frac{3}{4}\frac{D\alpha\nu r}{b\vartheta_o}$ in comparison with unity will be neglected throughout except, of course, the term $x - 1$. There is obtained:

$$\varphi_m - \alpha = \frac{1}{\chi}\left[\log\frac{b\vartheta_o}{\nu r} - \log\frac{3D\alpha(\sqrt{D+1}+1)}{8(\sqrt{D+1}-1)}\right.$$

$$\left.-2\sqrt{2D+1}\left(\tanh^{-1}\frac{1}{\sqrt{2D+1}} - \tanh^{-1}\frac{\sqrt{D+1}}{\sqrt{2D+1}}\right)\right]$$

The resistance coefficient and the Reynolds number are introduced as was done in the solution of the problem of the flow in a pipe. In analogy with the case of the pipe, let

$$C_f = \frac{\tau_o}{\frac{\rho u_m^2}{2}}$$

and

$$R_m = \frac{u_m R}{\nu}$$

where $R = r\vartheta_o$ is the local radius of the circular cross-sections of the diffuser. This gives

$$\varphi_m = \frac{u_m}{v_*} = \frac{\rho u_m}{\tau_o} = \frac{\sqrt{2}}{\sqrt{C_f}}$$

$$\frac{b\vartheta_o}{\nu r} = \frac{v_* r^2\vartheta_o}{\nu r} = \frac{Rv_*}{\nu} = \frac{R_m\sqrt{C_f}}{\sqrt{2}}$$

By substituting in the preceding expression and passing from hyperbolic functions to logarithms the resistance law for the diffuser is obtained:

$$
\left.\begin{aligned}
&\frac{1}{\sqrt{C_f}} = \frac{1}{\chi\sqrt{2}} \log \left(R_m \sqrt{C_f}\right) + A \\
&\text{where} \\
&A = \frac{\alpha}{\sqrt{2}} - \frac{1}{\chi\sqrt{2}}\left[\log\left(\frac{3\alpha\sqrt{2}}{8}\,\frac{\sqrt{D+1}+1}{\sqrt{D+1}-1}\,D\right)\right. \\
&\qquad \left. + \sqrt{2D+1}\,\log\frac{\left(1+\sqrt{2D+1}\right)\left(\sqrt{2D+1}-\sqrt{D+1}\right)}{\left(\sqrt{2D+1}-1\right)\left(\sqrt{2D+1}+\sqrt{D+1}\right)}\right]
\end{aligned}\right\} \tag{30}
$$

On passing from the diffuser to the straight pipe, that is, setting $D = 0$, there is obtained on evaluating the indeterminate expression by the rule of L'Hospital:

$$
\begin{aligned}
&\frac{1}{\sqrt{C_f}} = \frac{1}{\chi\sqrt{2}} \log \left(R_m \sqrt{C_f}\right) + A \\
&\text{where} \\
&A = \frac{\alpha}{\sqrt{2}} - \frac{1}{\chi\sqrt{2}} \log \frac{3\alpha\sqrt{2}}{4}
\end{aligned} \tag{31}
$$

As may be seen, the form of the resistance expression for the diffuser and that for the straight pipe is identical. It is very interesting that the diffuser parameter enters only into the makeup of the free term A and has no effect whatever on the terms containing the resistance coefficient and the Reynolds number. This is not entirely unexpected, because the resistance formula results from the shape of the velocity distribution curve near the outer boundary of the laminar layer, where the effect of the inertia forces is negligible and the mixing length, as in the case of the straight pipe, increases proportionally to the distance from the wall.

The numerical value of the free term A usually is determined experimentally for the pipe. As is known, the value of A is absolute, that is, independent of the Reynolds number, and is of the order of 4 to 5.[1]

---

[1]According to the tests of Nikuradse, $A = 4.88$; according to the present tests with the straight pipe $A = 4.01$.

There is every reason to expect (since the form of the expression for the resistance is the same as for the pipe) that the absoluteness of A extends also to the diffuser, while a variation is possible in the magnitude of α which for different values of the parameter D will be obtained as a solution of the transcendental equation

$$A = f(\alpha, D) = \text{constant}$$

The circumstance that the diffuser parameter D, which takes account of the effect of the inertia forces, does not enter the variable terms of the resistance formula leads to the conclusion that the form of the expression is not due to the approximation used above for the inertia integral. If it were possible to solve accurately the fundamental differential equation, the same form of expression would be obtained for the resistance formula.

The approximations assumed for the mixing length and inertia integral show up only on the free term. For this reason, no far reaching conclusions will be made with regard to the dependence of α on D [1] or physical explanation of this dependence, particularly since for practical computations not α but A is required, which in any case is obtained from experiment.

It is necessary to make one more remark with regard to the resistance formula. The Reynolds number entering the formula

$$Re_m = \frac{Ru_m}{\nu} = \frac{r\vartheta_o\ t_m}{r^2\nu} = \frac{\vartheta_o\ t_m}{r\nu}$$

is variable along the diffuser length. The friction coefficient, however, according to the assumption of radial flow should not vary over the diffuser length:

$$C_f = \frac{\tau_o}{\frac{1}{2}\rho u_m^2} = \frac{2f_o r^4}{r^4 \rho t_m^2} = \frac{2f_o}{\rho t_m^2}$$

This lack of correspondence is obtained as a result of the foregoing assumptions in considering the viscous friction at the wall which disturbs the radial character of the flow. The error thus obtained is vanishingly small. For small divergence angles of the diffuser the Reynolds number Re_m varies so little over the diffuser that C_f,

---

[1]It is seen, for example, since $D = f(Re)$ that also α in the diffuser will depend on the Reynolds number, that is, no longer be an absolute constant.

depending on the logarithm of Re_m, undergoes quite a negligible change.

II. EXPERIMENT AND COMPARISON WITH THEORY

In order to check all the assumptions made and results obtained in the preceding part, a series of tests were carried out on two conical diffusers with divergence angles (angle between axis and generator of cone) of 1° and 2°. In comparing the experimental with the theoretical results it was assumed that the comparison of the velocity and pressure distributions was of greatest importance. For this reason the experimental determination of such secondary factors as the frictional distribution over the cross section of the diffuser, the distribution of the mixing length, and so forth, were entirely omitted on the assumption that agreement or disagreement between theory and experiment for the main factors implies also the corresponding condition for the secondary factors.

(a) Test Setup

The diffusers on which the tests were conducted were of plywood construction having sheets, curved into conical segments, which were attached by wooden rings the internal diameters of which as far as possible were determined by accurate computation. The wooden rings were connected to each other by stringers disposed over the generators of the cone. The entry opening of the cone had a diameter of 240 millimeters and was connected by means of a transition piece with the cylindrical pipe previously investigated (reference 1) having a length of about 50 calibers. The diffusers were each 6 meters long. Behind the diffusers was placed a 2-meter section of a straight pipe having the diameter of the outlet section. This section was followed by a short accurate converging pipe that reconducted the flow into the pipe of 240-millimeter diameter. The latter pipe with the aid of two elbows was connected to the suction chamber. The inner surface of the diffuser and of the transition lengths was covered with shellac. The entire apparatus was suspended from the ceiling of a long passage of the laboratory. The lines in the diffuser cross sections along which the mean velocity distribution was measured lay in the horizontal plane passing through the axis of the apparatus. The first section was located in the center of the forward transition piece. The distance x, in meters, between the succeeding sections measured from the first is given in table I.

TABLE I

Section	1	2	3	4	5	6	7	8	9	10	11	12	13
x	0	1	2	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0

In distributing the sections along the diffuser it was assumed that over a certain portion beyond the transition pipe no typical diffuser flow will be observed to which the assumptions made in part I were applicable. Over this portion there will be a gradual transition from the conditions of flow in the straight pipe to the conditions of flow in the diffuser; that is, there will be a kind of diffuser entry length (Anlaufstrecke). On the basis of this fact, the main sections 3 to 13 were placed in the middle part of the diffuser. The test, as will be shown, justified these assumptions.

The velocity distributions were measured in sections 3 to 13 with the aid of the same pitot tube used in investigating the velocity distribution in the cylindrical pipe and the same coordinate apparatus (reference 1). Since the openings, through which the stem of the tube passed into the diffuser and came out through the opposite side, were made along the normal to the inner surface of the diffuser, the tube stem was slightly curved. It thus seems reasonable to assume that the velocity distribution was measured over the arcs of circles of greater radius, approximately corresponding to the arcs measuring the angle ϑ.

To measure the static pressure distribution along the diffuser, the usual system of brass tubes was mounted along the bottom stringer. The tubes were placed 200 millimeters apart from section 1 almost to the end of the diffuser.

(b) Tests

The velocity distribution in the chosen sections of the diffuser was measured for a single maximum discharge rate of air. The velocities were measured in each section from wall to wall, in order to be able to judge to some extent the axial symmetry of the velocity profiles. The points at which measurements of the velocity were made in each cross section were more often near the walls than near the diffuser axis. In testing, two micromanometers were read, one connected with the pitot tube (h_v), and the other a control (h_c) connected with a static tube placed in the cylindrical pipe ahead of the diffuser. After traversing each section of the velocity profile, the pitot tube was placed

on the diffuser axis and the air discharge was varied from zero to the maximum with the aid of the iris diaphragm at the chamber. Simultaneous readings were taken on the velocity (h_v) and control (h_c) manometers, and the magnitude was computed:

$$A = \frac{h_v}{h_c} \tag{32}$$

which for each section was plotted as a function of h_c. It was thus possible to compute the absolute value of the velocity at the diffuser axis as a function of the reading on the control manometer:

$$u_m = \sqrt{\frac{2}{\rho} F_v\, k_v\, \gamma \xi_T\, A\, h_c}$$

where

$$\rho = 0.125\, \frac{288\, p_{atm}}{760(273 + t)}$$

is the density of the air with corrections on the temperature of the air and the atmospheric pressure;

F_v and k_v sine of angle of inclination of manometer tube and coefficient of manometer, respectively

ξ_T coefficient of pitot tube

γ density of spirit in manometer

Thus, by instantaneous reading of two manometers, it was possible to determine the velocity at any point of any section as a fraction of the velocity at the diffuser axis in the same section:

$$\frac{u}{u_m} = \sqrt{\frac{h_v}{Ah_c}}$$

In determining the velocity distribution by this formula, near the diffuser walls a correction was made for the effect of the walls on the readings of the pitot tube. It is known that the presence of a wall near the stem of the tube gives rise to a certain increase in the velocity between the stem of the tube and the wall, and this leads to a drop in the static pressure and results in an increase in the manometer readings above the true values. The correction was applied by multiplying the radical on the right side of this formula by the correction coefficient ζ. The dependence of this coefficient on the distance y from the wall to the center of the front opening of the tube is shown on figure 3. This curve was obtained from tests in the straight pipe (reference 1) where the velocity distribution near the wall was measured both by the pitot tube and the total pressure micro nozzle, on the readings of which the effect of the wall was vanishingly small. The test points on figure 3 were obtained for various Reynolds numbers. As may be seen, with increasing distance from the wall the values of ζ rapidly approach unity.

In addition to the correction near the wall, a correction was made for the change in diffuser cross section caused by the stem of the tube in determining the absolute value of the velocity u_m on the diffuser axis. This correction coefficient μ was determined as:

$$\mu = \frac{S - S_{mp}}{S}$$

where $S = \pi r^2 \vartheta_o^2$ circular area of the diffuser cross section, $S_{mp} = 2r\vartheta_o d$ part of diffuser cross section taken up by stem of tube (d diameter of tube). The justification for applying the correction by this method is given in a paper on tests on a pipe (reference 1).

The distribution of the velocity ratio $\frac{u}{u_m}$ for the various sections of the diffusers of 1° and 2° is shown in figures 4 and 5, where the magnitude

$$\xi = \frac{\vartheta}{\vartheta_o} = \frac{z}{z_o}$$

is laid off on the axis of abscissas. The values z and z_o, the lengths of the arcs of the circles having their center at the vertex of the cone and passing through each section, were read off directly with the coordinate apparatus by divisions carried on the stem of the pitot

tube. On these figures the velocity curves are shifted upward with respect to each other. The dots and circles denote the measurements at the right and left halves of the velocity profile.

In examining these curves there is observed first of all that: (1) The symmetry of the profile with respect to a vertical plane is sufficiently good (the dots and circles are not very far removed from the mean curve), (2) the assumption in part I on the similarity of the velocity profiles for sections 3 to 13 is observed to be well satisfied. An exception occurs in the case of the 1° diffuser for section 5, which, for an unexpected reason, drops out of the general series.

Of interest on the above curves are the velocity distributions for sections 1 and 2. The latter distributions (especially for the 2° diffuser) clearly indicate the transition region from the straight pipe to the diffuser. Here, it should be remarked that the velocity distribution in section 1 is not characteristic for the pipe and is also transitional, since it must be supposed that the effect of the diffuser extends somewhat ahead into the pipe. On the basis of the assumption that the shape of the velocity profile is entirely determined by the static pressure gradient, it may be said that the region of influence of the diffuser on the pipe will extend into the latter up to a point where the pressure gradient in magnitude and sign will agree with the corresponding value for the pipe.

After measuring the velocity profiles over the diffuser sections, pressure distribution measurements were carried out with the aid of the above-mentioned system of static tubes. The measurements were made for five discharge rates for the 1° diffuser and eight discharge rates for the 2° diffuser.

The results are given on figures 6 and 7, where the lowering in pressure in kilograms per square meter is plotted as a function of the distance from section 1. As may be seen, in both diffusers there was a positive pressure gradient. The static pressure distribution transverse to the diffuser was also measured. No pressure gradient with respect to ϑ was observed. This completed the test program.

(c) Checking of Fundamental Assumptions

The confirmation of the fundamental assumption made in constructing the theory, namely, the assumption of radial flow will be discussed first. On applying this assumption to the equation of continuity, the result (formula 5) that the product ur^2 should not be a function of r was obtained. The values of this product on the diffuser axis where $u = u_m$ will be computed. In tables 2 and 3 are given the values of u_m

and r for both diffusers for maximum discharge rate, and the products are computed. The values of u_m for the various sections of each diffuser, with the aid of the curve $A = f(h_c)$, was reduced to a single reading of the control manometer, that is, to a single discharge rate.

TABLE II.- VALUES OF r, u_m AND $u_m r^2$ FOR 1° DIFFUSER

Section	3	4	5	6	7	8	9	10	11	12	13
r (m)	15.73	15.93	16.13	16.33	16.53	16.73	16.93	17.13	17.33	17.53	17.73
u_m (m/sec)	21.0	20.2	19.9	19.75	19.1	18.7	18.5	18.05	17.70	17.34	16.9
$u_m r^2$	5200	5130	5190	5270	5225	5225	5310	5300	5320	5340	5310

TABLE III.- VALUES OF r, u_m AND $u_m r^2$ FOR 2° DIFFUSER

Section	3	4	5	6	7	8	9	10	11	12	13
r (m)	8.88	9.08	9.28	9.48	9.68	9.88	10.08	10.28	10.48	10.68	10.88
u_m (m/sec)	18.30	17.80	17.18	16.65	16.18	15.50	14.95	14.50	13.66	13.17	12.63
$u_m r^2$	1443	1470	1480	1500	1518	1515	1525	1540	1500	1500	1500

For greater clearness the values of the product $u_m r^2$ are plotted as a function of x in figures 8 and 9. As may be seen, the product $u_m r^2$ does not remain constant but increases in a regular manner along the axes of the diffusers. The regular character of the increase in $u_m r^2$ indicates that the obtained result is not due to experimental error but to a certain regular disturbance in the radial character of the flow that arises from the effect of the viscosity. The difference

between the maximum and minimum values of $u_m r^2$ is, however, sufficiently small and does not exceed 4 percent for the 1° diffuser and 7 percent for the 2° diffuser. The regularity of the increase in $u_m r^2$ is broken only in the last sections of the 2° diffuser, where evidently some other unaccounted-for effect enters. It is possible that the irregularity at these points may be due to some error in the test setup.

Thus the radiality of the flow at the diffuser axis (and therefore also over the entire region of the flow) is satisfied only with a certain, though rather large, degree of approximation. A more detailed discussion on the radiality of the flow will be given in part III of this paper. For the present, however, the obtained increasing values $u_m r^2 = f(r)$ will be replaced by mean constant values $(u_m r^2)_m$. On figures 10 and 11 the values of $(u_m r^2)_m$ are plotted and the mean values indicated for various discharge rates for both diffusers.

Another method of checking the radiality of the flow is to study the increase in the static pressure along the diffuser. In the first part the result was obtained that in the case of radial flow the generalized pressure $\overline{p} - \tau_{rr}$ should vary along the diffuser according to the law

$$p_1 = -\frac{\rho G}{4}\frac{1}{r^4} + \left(p_{1_0} - \frac{\rho G}{4r_0^4}\right) \qquad \text{(formula 8)}$$

At the wall, that is, for $\xi = 1$, the pulsations should die down $(\tau_{rr} = 0)$ and $p_1 = \overline{p}$. Thus, in the case of the correctness of the foregoing law of increase in pressure, there should in plotting $\overline{p}$ as a function of $\frac{1}{r^4}$ be obtained straight lines, the slope of which is determined by the value of G:

$$G = -\frac{4}{\rho}\frac{d\overline{p}}{d(r^{-4})} \tag{33}$$

This plot for the two diffusers is shown on figures 12 and 13. As may be seen, the law expressed by formula (8) is excellently confirmed. The various straight lines on these figures correspond to the values $(u_m r^2)_m$ on the previous figures. For each straight line is indicated the value of G determined by formula (33).

In measuring the pressure distribution for the 2° diffuser use was made of a more improved system of manometers as compared with that used for measuring the pressure in the 1° diffuser. Because of this it was possible in figure 13 to plot the test points directly, while in plotting figure 12 the values of the pressure had to be taken from the comparison curves (fig. 6). This explains the greater scatter of the points on figure 13 as compared with figure 12.

On these curves the values of x corresponding to the sections at which the velocity distribution was determined are indicated below. It is of interest that the pressure variation law is confirmed over a considerably greater length than the law of similarity of velocity profiles. This should also be the case since according to equation (11) the value of the magnitude G characterizing the change in pressure depends not directly on the velocity distribution but on a certain integral of this distribution which should not be very sensitive to a change in the velocity profile.

(d) Check of the Velocity and Resistance Laws

On figures 14 and 15 are given the comparison of the experimental and theoretical profiles $\frac{u}{u_m} = f(\xi)$ of the velocities determined by formula (26) for the maximum Reynolds numbers attainable.[1] The value of the diffuser parameter D for the computations by formula (26) was determined to a first approximation by the formula

$$D = \vartheta_0 \varphi_m^2 (1 - 2I_0)$$

where the value of I_0 was computed with the aid of graphical integration of the mean experimental velocity profile. After obtaining the theoretical profiles shown on figure 14 and 15, the value of D was corrected by determining I_0 from the computed theoretical profiles. The second approximation for D differed so slightly from the first that further approximations of the velocity profile seemed superfluous. The value of φ_m entering the formula for D was determined as

$$\varphi_m = \frac{\sqrt{2}}{\sqrt{C_f}}$$

[1]Note the transition here from the values $\varphi_m - \varphi$ to the values $\frac{u}{u_m}$, determining φ_m according to the resistance law.

from the derived resistance law (30), where the value of the absolute constants χ and A, on the basis of tests with cylindrical pipes, were taken as

$$\chi = 0.434$$

and

$$A = 4.01$$

The Reynolds number Re_m was determined as the mean of the Reynolds numbers

$$Re_m = \frac{r\vartheta_o u_m}{\nu} = \frac{\vartheta_o (u_m r^2)_m}{r\nu}$$

over the diffuser sections. The possibility of this averaging was justified by the fact that the change in Re_m along the diffuser practically had no effect on the values of C_f and φ_m.

The final values of D and φ_m for both diffusers for mean Reynolds numbers were obtained as:

For the 1° diffuser $Re_{m\ av} = 178,000$ $\varphi_m = 26.65$ $D = 2.27$

For the 2° diffuser $Re_{m\ av} = 177,000$ $\varphi_m = 26.55$ $D = 5.56$

According to figures 14 and 15 it may be stated that the theoretical formula satisfactorily agrees with the test results. Some deviation is observed in the middle portion of the curve where the test points have a tendency to drop below somewhat. The reason for this deviation must be sought apparently in the fact that the actual distribution of the mixing length in the diffuser differs somewhat from the cubical parabola assumed.

To illustrate the degree of accuracy of the solution of the fundamental differential equation by approximating the inertia integral by the exponential relation

$$I = \frac{1}{2} + \left(I_o - \frac{1}{2} \right) \xi^{\frac{3}{2}}$$

figures 16 and 17 are presented, where the continuous curve gives the above integral and the points give the results of computing I by means of the graphical integration of the obtained theoretical curve of the velocity distribution. The discrepancies between these values of I are so small that it may be affirmed that the degree of accuracy of the solution of the fundamental differential equation by this approximation is sufficiently high.

the velocity distribution. The discrepancies between these values of I are so small that it may be affirmed that the degree of accuracy of the solution of the fundamental differential equation by this approximation is sufficiently high.

It is very unfortunate that formula (26) does not enable the analytical computation of the values of the inertia integral, since the problem reduces to incomputable quadratures. Otherwise by expressing I_o in terms of D by means of the above integration the dependence of D could be found only on the Reynolds number and the divergence angle of the diffuser.

As an illustration of formula (27) of the velocity distribution for the straight pipe (zero diffuser), figure 18 shows the results of the tests in the pipe as given (reference 1) and, as may be seen, formula (27), for the values of χ chosen on the basis of the resistance law, is in excellent agreement with the test results. It is interesting to note that the curve representing formula (27) lies very near the curve drawn according to the theory of Mattioli, which until now of all semiempirical theories best agrees with experiment. This consideration is of great importance since formula (27) is extremely simple; whereas the theory of Mattioli does not lead to any finite formula at all, and to draw the velocity distribution it is necessary to carry out a graphical integration.

For checking the resistance formula in the usual manner by comparing the theoretical and experimental laws of variation of the resistance coefficient with the Reynolds number, it would have been necessary for to find the experimental values of the frictional shear at the diffuser wall. It was considered illogical to carry out the above operation, since it would then be necessary again to make use of the theoretical equation of motion that gives the relation between the friction, the inertia forces, and the pressure gradient. For this reason it was considered of greater interest to compare the theoretical value of the magnitude G, characterizing the increase in static pressure along the diffuser, with the corresponding test values obtained by measuring the slopes of the curves on figures 12 and 13.

The theoretical value of G may be obtained from equation (11):

$$G = \frac{2b^2}{\vartheta_o}\left(2\vartheta_o \int_0^1 \varphi^2 \xi \, d\xi - 1 \right) = 2b^2\varphi^2{}_m \left(2I_o - \frac{1}{\vartheta\varphi^2{}_m} \right)$$

$$= 2t^2{}_m \left(2I_o - \frac{1}{\vartheta_o\varphi^2{}_m} \right) \qquad (34)$$

where t_m is no other than the value $(u_m r^2)_m$, depending on the rate of discharge of the fluid through the diffuser, that is, on the Reynolds number

$$(u_m r^2)_m = \left(\frac{Re_m r \nu}{\vartheta_0} \right)_m$$

By integrating the theoretical velocity profile, I_0 is determined; φ_m is determined as a function of the Reynolds number according to the resistance formula.

Thus, expression (34) is the theoretical dependence of G on the Reynolds number while expression (33), which determines G by the slopes of the test curves, is the corresponding experimental dependence.

On figures 19 and 20 are given the curves of the theoretical and experimental relations $G = f(t_m)$. As may be seen, in both cases the test points lie below the theoretical curves. Part III of the present paper will show more in detail the reason for the obtained (not very large, it is true) discrepancies in the values of G.

In concluding this part, the theoretical curves of the friction distribution transverse to and along the diffuser are given. By formula (10):

$$\frac{\tau}{\tau_0} = \frac{1}{\xi}\left(2\vartheta_0 \int_0^{\xi} \varphi^2 \xi d\xi - \frac{\vartheta_0 G \xi^2}{2b^2} \right) = \xi \left(2\vartheta_0 J - \frac{\vartheta_0 G}{2b^2} \right)$$

By substitution, according to expression (11),

$$\frac{\vartheta_0 G}{2b^2} = 2\vartheta_0 J_0 - 1$$

there is obtained

$$\frac{\tau}{\tau_0} = \xi \left[2\vartheta_0 (J - J_0) + 1 \right]$$

According to the assumed approximation for the inertia integral,

$$\frac{\tau}{\tau_o} = \xi\left[\vartheta_o\left(\varphi_m^2 - 2J_o\right)\left(1 - \xi^{\frac{3}{2}}\right) + 1\right]$$

By substituting $J_o = \varphi^2{}_m I_o$ And introducing the diffuser parameter D there is obtained finally

$$\frac{\tau}{\tau_o} = \xi\left[D\left(1 - \xi^{\frac{3}{2}}\right) + 1\right] \tag{35}$$

On figure 21 are constructed the curves $\frac{\tau}{\tau_o} = f(\xi)$, according to formula (35), for the diffusers and cylindrical pipe. It is interesting how such small diffuser divergences as 1^o and 2^o, which seem insignificant at first sight, affect the character of the friction distribution. As may be seen, near $\xi = 0.6$ the local frictional shear for the 1^o diffuser is approximately twice and for the 2^o diffuser about four times the value for the straight pipe.

On figures 22 and 23 are drawn the friction distributions along the walls of the diffusers according to the law

$$\tau_o = \frac{f_o}{r^4}$$

for various values of $t_m = (u_m r^2)_m$. The values of f_o is expressed as

$$f_o - \frac{1}{2} C_f \rho t^2{}_m = \rho\left(\frac{t_m}{\varphi_m}\right)^2$$

III. SOME CONSIDERATIONS WITH REGARD TO THE DEVIATIONS FROM THE RADIAL FLOW ASSUMPTION

It has already been pointed out that the nonconstancy of the values of $u_m r^2$ along the diffuser axis indicates a systematic deviation from the assumed radial flow. It is true that the deviation of the flow from the radial direction is extremely small. It is interesting,

however, to estimate to a first approximation what these deviations are and their effect on certain final results.

Investigation of the deviation from the radiality is possible by considering the so-called "secondary flows," that is, the transverse components v of the mean velocity directed along the circular arcs measuring the angles ϑ. The direct measurement of these velocities is, of course, impossible.

An attempt was made to use the continuity equation connecting the velocities v with the velocities u by analyzing with its aid the test curves of the velocity distribution. This attempt did not, however, lead to any good results, for it was found that the accuracy of the test for this purpose was insufficient. Some conclusions may, however, be drawn from the obtained experimental law of the variation of the magnitude $u_m r^2$ along the axis.

It is assumed that at any point M with coordinates ϑ and r within the diffuser there are two component velocities u and v (fig. 24). By adding u and v vectorially there is obtained the modulus of the resultant velocity

$$w = \sqrt{u^2 + v^2}$$

By prolonging the direction of w to its intersection with the diffuser axis at point N, the point N may be approximately considered as a certain fictitious source producing a radial velocity field near point M. The distance of this fictitious source is denoted from the vertex of the cone by $ON = \Delta r$. By proceeding in the same way for each point within the diffuser, various values will naturally be obtained for Δr. In the general case,

$$\Delta r = f(r, \vartheta, u, v)$$

The extent by which Δr differs from zero will characterize the degree of nonradiality of the true motion.

This function is expressed through the coordinates of the point and the values of the velocities. From triangle ONM there is obtained (fig. 24)

$$\vartheta_1 = \vartheta + \tan^{-1} \frac{v}{u}$$

On account of the smallness of v it may be assumed that $\tan^{-1}\frac{v}{u} \approx \frac{v}{u}$. Bearing in mind the smallness of the angles, write

$$(r - \Delta r)\vartheta_1 = r\vartheta$$

whence, combining with the above equation gives

$$\Delta r = \frac{r\dfrac{v}{\vartheta}}{u + \dfrac{v}{\vartheta}} \tag{36}$$

The limit which this function approaches will be found as ϑ approaches zero, that is, the value of this function on the axis of the diffuser:

$$\Delta r_m = \lim_{\vartheta \to 0} \Delta r = \frac{r \lim\limits_{\vartheta \to 0}\left(\dfrac{v}{\vartheta}\right)}{u_m + \lim\limits_{\vartheta \to 0}\left(\dfrac{v}{\vartheta}\right)}$$

From considerations of symmetry $v = 0$ on the diffuser axis. The limit of the ratio $\frac{v}{\vartheta}$ will be found by making use of the equation of continuity, which for the two velocity components in spherical coordinates, will have the form:

$$\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \vartheta} + \frac{2u}{r} + \frac{v}{r\vartheta} = 0$$

Combining the first and third terms of the equation results in

$$\frac{\partial(ur^2)}{\partial r} + r\frac{\partial v}{\partial \vartheta} + r\frac{v}{\vartheta} = 0$$

whence the required ratio

$$\frac{v}{\vartheta} = -\left(\frac{1}{r}\ \frac{\partial(ur^2)}{\partial r} + \frac{\partial v}{\partial \vartheta}\right)$$

On passing to the limit there is obtained:

$$\lim_{\vartheta \to 0}\left(\frac{v}{\vartheta}\right) = -\frac{1}{r}\ \frac{\partial(u_m r^2)}{\partial r} - \lim_{\vartheta \to 0}\left(\frac{\partial v}{\partial \vartheta}\right) \tag{37}$$

On the other hand, from L'Hospital's rule there is obtained:

$$\lim_{\vartheta \to 0}\left(\frac{v}{\vartheta}\right) = \lim_{\vartheta \to 0}\left(\frac{\partial v}{\partial \vartheta}\right) \tag{37a}$$

Substituting this in expression (37) and solving the obtained expression for the limit of $\frac{v}{\vartheta}$ gives:

$$\lim_{\vartheta \to 0}\left(\frac{v}{\vartheta}\right) = -\frac{1}{2r}\ \frac{\partial(u_m r^2)}{\partial r}$$

Thus,

$$\Delta r_m = -\frac{\dfrac{\partial(u_m r^2)}{\partial r}}{2u_m - \dfrac{1}{r}\ \dfrac{\partial(u_m r^2)}{\partial r}} \tag{38}$$

The value of the derivative $\frac{\partial(u_m r^2)}{\partial r}$ is experimentally determined from the curves of figures 8 and 9. The curve $u_m r^2 = f(r)$ shown on these figures may, with a very large degree of accuracy, be represented as a linear function:

$$u_m r^2 = ar + b$$

In determining the values of a and b, use was made of the method of least squares, which for the given case gives the generally known formulas:

$$a = \frac{\begin{vmatrix} \sum\limits_n u_m r^3 & \sum\limits_n r \\ \sum\limits_n u_m r^2 & n \end{vmatrix}}{\begin{vmatrix} \sum\limits_n r^2 & \sum\limits_n r \\ \sum\limits_n r & n \end{vmatrix}}; \qquad b = \frac{\begin{vmatrix} \sum\limits_n r^2 & \sum\limits_n u_m r^3 \\ \sum\limits_n r & \sum\limits_n u_m r^2 \end{vmatrix}}{\begin{vmatrix} \sum\limits_n r^2 & \sum\limits_n r \\ \sum\limits_n r & n \end{vmatrix}},$$

where the sums are taken for n readings of $u_m = f(r)$. (In the given case n is the number of diffuser sections in which u_m was measured.) For these cases there was obtained (for maximum Reynolds numbers):

1° diffuser	a = 88.6 square meters per second
	b = 3769 cubic meters per second
2° diffuser	a = 59.6 square meters per second
	b = 923.7 cubic meters per second

This leads to the computation formula:

$$\Delta r_m = -\frac{a}{2u_m - \frac{a}{r}}. \tag{39}$$

From expression (37a) it is found that the first derivative of the curve of transverse velocity distribution v with respect to ϑ on the axis will be:

$$\left(\frac{\partial v}{\partial \vartheta}\right)_{\vartheta=0} = -\frac{a}{2r}.$$

where the minus sign indicates that the velocities v are in the direction opposite to those shown on figure 24, that is, from the wall to the axis.

In tables IV and V are given the computed values of the function $\Delta r_m = f(r)$ according to formula (39).

TABLE IV.– COMPUTATION OF Δr_m FOR 1° DIFFUSER

Section	3	4	5	6	7	8	9	10	11	12	13
r (m)	15.73	15.93	16.13	16.33	16.53	16.73	16.93	17.13	17.33	17.53	17.73
u_m (m/sec)	21.0	20.2	19.9	19.75	19.1	18.7	18.5	18.05	17.70	17.34	16.9
Δr_m	-2.44	-2.55	-2.58	-2.60	-2.70	-2.76	-2.79	-2.86	-2.92	-2.99	-3.07

TABLE V.– COMPUTATION OF Δr_m FOR 2° DIFFUSER

Section	3	4	5	6	7	8	9	10
r (m)	8.88	9.08	9.28	9.48	9.68	9.88	10.08	10.28
u_m (m/sec)	18.30	17.80	17.18	16.65	16.18	15.50	14.95	14.50
Δr_m	-2.00	-2.06	-2.13	-2.20	-2.28	-2.40	-2.49	-2.57

The minus sign before the values of Δr_m indicates that the fictitious source is located not ahead of the vertex of the cone but behind it. As may be seen, the values of Δr_m for the 1° diffuser constitute on the average 16 to 17 percent of the values of r and for the 2° diffuser 23 to 25 percent of the corresponding values of r. This, as may also

be expected, is a consequence of the fact that the deviation from radiality for the 2° diffuser is greater than for the 1° diffuser.

Moreover, it is seen that Δr_m varies over the diffuser length. The variations of Δr_m, it is true, are not large in comparison with r and constitute over the entire range investigated 3.5 to 4 percent for the 1° diffuser and 5.5 to 6.5 percent for the 2° diffuser. The latter circumstance makes it possible to assume approximately that near the axis flow may actually be thought of as originating at the fictitious source behind the cone vertices at the mean distances

$$\overline{\Delta r_m} = -2.75 \text{ for the } 1^\circ \text{ diffuser}$$

and

$$\overline{\Delta r_m} = -2.27 \text{ for the } 2^\circ \text{ diffuser}$$

Since the true nonradial flow near the axis is approximately replaced by a fictitious radial flow originating at the second source, the equation of continuity valid for radial flow may be applied to it; that is, write

$$\frac{\partial}{\partial r}\left[u_m\left(r - \overline{\Delta r_m}\right)^2\right] = 0$$

whence

$$u_m\left(r - \overline{\Delta r_m}\right)^2 = t_m = \text{constant} \tag{40}$$

The correctness of this expression is readily proved by drawing curves similar to those of figures 8, 9, 10, and 11, the values $u_m(r - \Delta r_m)^2$ being laid off along the axis of ordinates. These curves are given on figures 25 and 26. On comparing with those of figures 10 and 11, it would seem that the constancy of t_m determined by formula (40) is considerably better satisfied than without taking the displacement of the source into account. In drawing these curves $\overline{\Delta r_m}$ was considered as independent of the velocity on the diffuser axis, that is, independent of the Reynolds number. This evidently corresponds with the facts, since in determining Δr_m, formula (38), both the numerator and the denominator may with a very large approximation be assumed proportional to the velocity u_m.

For still greater assurance as regards the possibility of representing the nonradial flow as a radial flow originating from a

fictitious source, $\overline{\Delta r_m}$ was determined by still another method. Let it be assumed that there is given a test distribution along the diffuser axis of the magnitude $u_m r^2$. It may be asked what increment Δr_m must be given to r in order that the value of the product $u_m(r - \overline{\Delta r_m})^2$ will deviate least from any constant values for each Reynolds number. The statement of the problem in this form leads to finding a minimum of the function:

$$\Phi = \sum_n \left[\frac{1}{n}\sum_n u_m (r - \overline{\Delta r_m})^2 - u_m (r - \overline{\Delta r_m})^2\right]^2,$$

where the first part represents the sum of the squares of the deviations of the values $u_m(r - \overline{\Delta r_m})^2$ from the mean taken for n readings. Removing parentheses, differentiating with respect to $\overline{\Delta r_m}$, and equating $\frac{d\Phi}{d\,\overline{\Delta r_m}}$ to zero yields, after transforming the cubic equation in $\overline{\Delta r_m}$:

$$\left[\frac{1}{n}\Big(\sum_n u_m\Big)^2 - \sum_n u_m^2\right]\overline{\Delta r_m}^3 + \left[\sum_n u_m^2 r - \frac{1}{n}\sum_n u_m \sum_n u_m r\right]\overline{\Delta r_m}^2 -$$

$$-\left[3\sum_n u_m^2 r^2 - \frac{1}{n}\sum_n u_m r^2 \sum_n u_m - \frac{2}{n}\Big(\sum_n u_m r\Big)^2\right]\overline{\Delta r_m} +$$

$$+\sum_n u_m^2 r^3 - \frac{1}{n}\sum_n u_m r^2 \sum_n u_m r = 0.$$

The computations according to this equation, conducted with very great accuracy, gave for the diffusers for maximum discharge rate the roots:

1° diffuser $\overline{\Delta r}_m = -2.44$ meters

2° diffuser $\overline{\Delta r}_m = -2.20$ meters

The good agreement of the obtained values of $\overline{\Delta r}_m$ with those computed from the continuity equation very obviously indicates the acceptability of all the foregoing assumptions with regard to Δr.

It is very interesting to inquire whether the results of the present investigation conducted for the region near the diffuser may be extended to the entire region of flow in the diffuser, assuming, of course, in the general case $\Delta r = f(\vartheta)$. That Δr necessarily must depend on ϑ follows from its very definition. Thus, for example, on the basis of expression (36) at the wall, that is, for $\vartheta = \vartheta_o$ it it follows that:

$$\lim_{\vartheta \to \vartheta_o} \Delta r = -\frac{r}{\vartheta_o \lim\limits_{\substack{u \to o \\ v \to o}} \frac{u}{v} + 1}$$

where

$$\lim_{\substack{v \to o \\ u \to o}} \frac{u}{v} = \lim_{\substack{u \to o \\ v \to o}} \frac{\frac{\partial u}{\partial \vartheta}}{\frac{\partial v}{\partial \vartheta}} = -\infty$$

since on the basis of the equation of continuity

$$\lim_{\substack{u \to o \\ v \to o}} \frac{\partial v}{\partial \vartheta} = \lim_{\substack{u \to o \\ v \to o}} \left[-\frac{1}{r} \frac{\partial ur^2}{\partial r} - \frac{v}{\vartheta} \right] = 0$$

while the value of $\frac{\partial u}{\partial \vartheta}$ at the wall is, of course, not equal to zero. Thus,

$$(\Delta r)_{\vartheta = \vartheta_o} = 0$$

The function $\Delta r = f(\vartheta)$ evidently should be of the following form: It should increase sharply over the thickness of the laminar sublayer (where the effect of the viscosity, as has been seen, particularly disturbs the radiality of the flow), and attain somewhere near the edge of the laminar sublayer a maximum followed by a smooth drop in the direction toward the diffuser axis to the foregoing numerical values.

The possibility of representing the entire flow in the diffuser as radial with displaced source has, in the writer's opinion, a certain practical value. It is possible thus, with a very high degree of approximation, taking for the continuity equation the expression

$$\frac{\partial}{\partial r}\left[u_m\left(r - \Delta r\right)^2\right] = 0$$

to consider the phenomenon as subject to the equations of motion applicable to the radial flow but substitute in them throughout $r - \Delta r$ in place of r. Thus there may be written instead of formula (6)

$$G = \frac{1}{\rho}\,\frac{\partial p_1}{\partial(r - \Delta r)}\,(r - \Delta r)^5 \tag{6'}$$

and

$$f = \tau(r - \Delta r)^4$$

In place of expression (8) there is obtained:

$$p_1 - p_{1_o} = -\frac{1}{4}\,\rho G\left[\frac{1}{(r - \Delta r)^4} - \frac{1}{(r - \Delta r)_o^{\,4}}\right] \tag{8'}$$

Expressions (33) and (34) determining the experimental and theoretical values of G become, respectively,

$$G = -\frac{4}{\rho}\,\frac{dp}{d\left[\left(r - \overline{\Delta r}_m\right)^{-4}\right]} \tag{33'}$$

$$G = 2\left[u_m\left(r - \overline{\Delta r_m}\right)^2\right]^2\left[2I_o - \frac{1}{\vartheta_o\varphi_m^2}\right] \qquad (34')$$

These formulas were written for the axis of the diffuser where $\Delta r = \Delta r_m$, still retaining the previously made assumption that

$$\frac{\partial p}{\partial \vartheta} = 0 \quad \text{and} \quad \frac{\partial G}{\partial \vartheta} = 0$$

An attempt will be made to verify whether the agreement of the theoretical and experimental values of G is improved by the correction for the nonradiality according to formulas (33') and (34'). On figures 27 and 28, analogous to figures 12 and 13, are drawn the curves

$$p = f\left[\frac{1}{\left(r - \overline{\Delta r_m}\right)^2}\right]$$

As may be seen, the introduction of the correction Δr_m has almost no effect on the appearance of the above curves.

On figures 29 and 30, a comparison is made between the theoretical and experimental values of G determined by formulas (33') and (34'). As may be seen, comparing these curves with the previous figures 19 and 20, it may be found that although the correction $\overline{\Delta r_m}$ gave excellent results for the 1° diffuser, the test points for the 2° diffuser lie on the other side of the theoretical curve and at a greater distance than on figure 20.

It would seem that it is not difficult to establish the reason for the result for the 2° diffuser. Up to now it has been assumed that the gradient of τ_{rr} along r is negligible in comparison with the corresponding gradient of p, and this permitted consideration of the value of G as independent of ϑ. Evidently, although for the 1° diffuser this neglect was justifiable; for the 2° diffuser, where the intensity of the turbulence fluctuations should be higher, it is necessary to introduce a correction for τ_{rr} which is, of course, a function of both ϑ and r. The theory is as yet unable to contribute anything in this direction.

CONCLUSIONS

The theoretical and test results presented on the problem of the turbulent motion in conical diffusers, are, of course, to a large extent approximate. The obtained satisfactory agreement between the theoretical and the test results is a consequence of the smallness of the divergence angles of the diffusers investigated. Undoubtedly, with further increase in the divergence the effect of the secondary flows will show up to an increasingly greater extent and finally will lead to separation of the flow. The investigation of a separated flow in the diffuser is of great theoretical and practical interest. There does not as yet appear any way leading to the solution of this problem in the full sense of the word: that is, a solution which is self-contained with respect to the boundary conditions. In the solution of such a problem further development of this method of considering the flow as emanating from a fictitious source may be of value.

Noted here are a few of the most important conclusions drawn from the present investigation:

1. The assumption made at the beginning of the investigation of the radiality of the flow holds true to a satisfactory degree for both diffusers investigated.

2. The assumption of the absolute character of the curve of mixing length for the straight pipe and diffusers and the representation of this curve by a cubical parabola give good agreement of the obtained velocity profiles with experiment.

3. The resistance formula for the diffusers is identical with that for the pipe.

4. The values of the absolute turbulence constants χ and A in the resistance formula determined on the basis of tests on the straight pipe are applicable also to diffusers, and this confirms the absolute character of the constants.

5. The approximate representation of the acutally nonradial flow in a diffuser by a radial flow originating from a fictitious source gives the necessary correction in computing the velocity drop along the axes of the 1° and 2° diffusers and in computing the pressure drop along the 1° diffuser. In computing the pressure drop for the 2° diffuser, less favorable results were obtained because sufficient account was not taken of the magnitude connecting the normal component of the turbulent stresses with the gradient along the diffuser. This magnitude cannot as yet be theoretically obtained.

In conclusion, the writer wishes to express his deep appreciation to P. E. Kuryatnikov for assisting in the tests and computations in connection with the present paper.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

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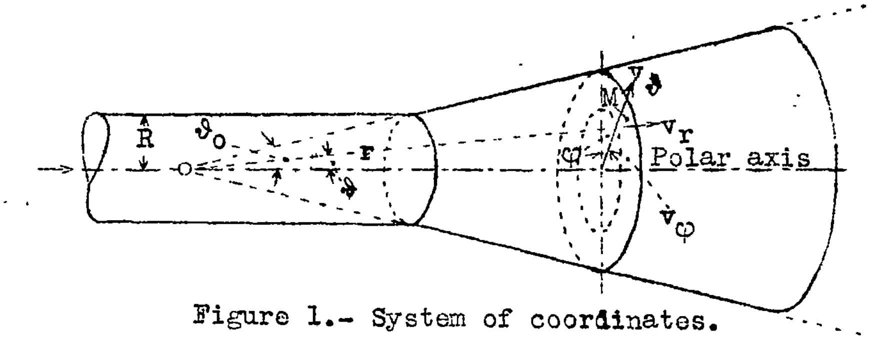


Figure 1.- System of coordinates.

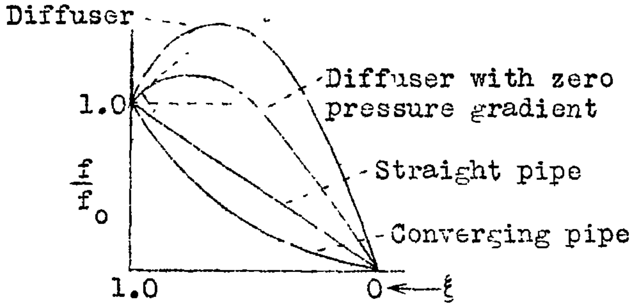


Figure 2.- Friction distribution over a diverging, straight and converging pipe.

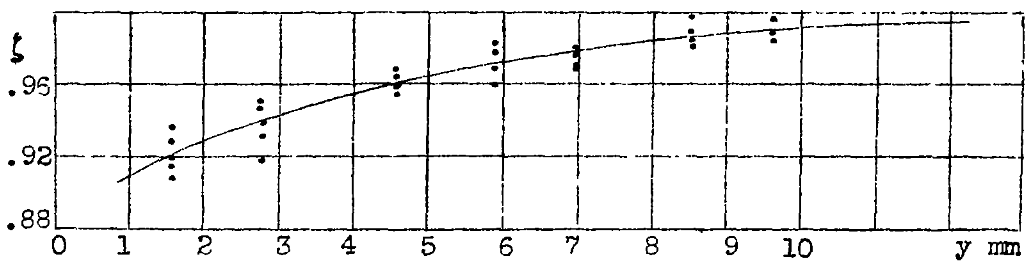


Figure 3.- Correction coefficient ζ, taking account of the effect of the wall on the Pitot tube reading, plotted as a function of the distance from the wall.

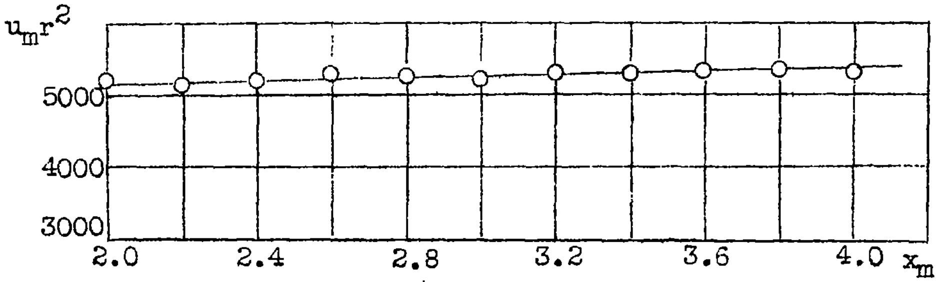


Figure 8.- Variation of $u_m r^2$ along axis of 1° diffuser for maximum discharge rate.

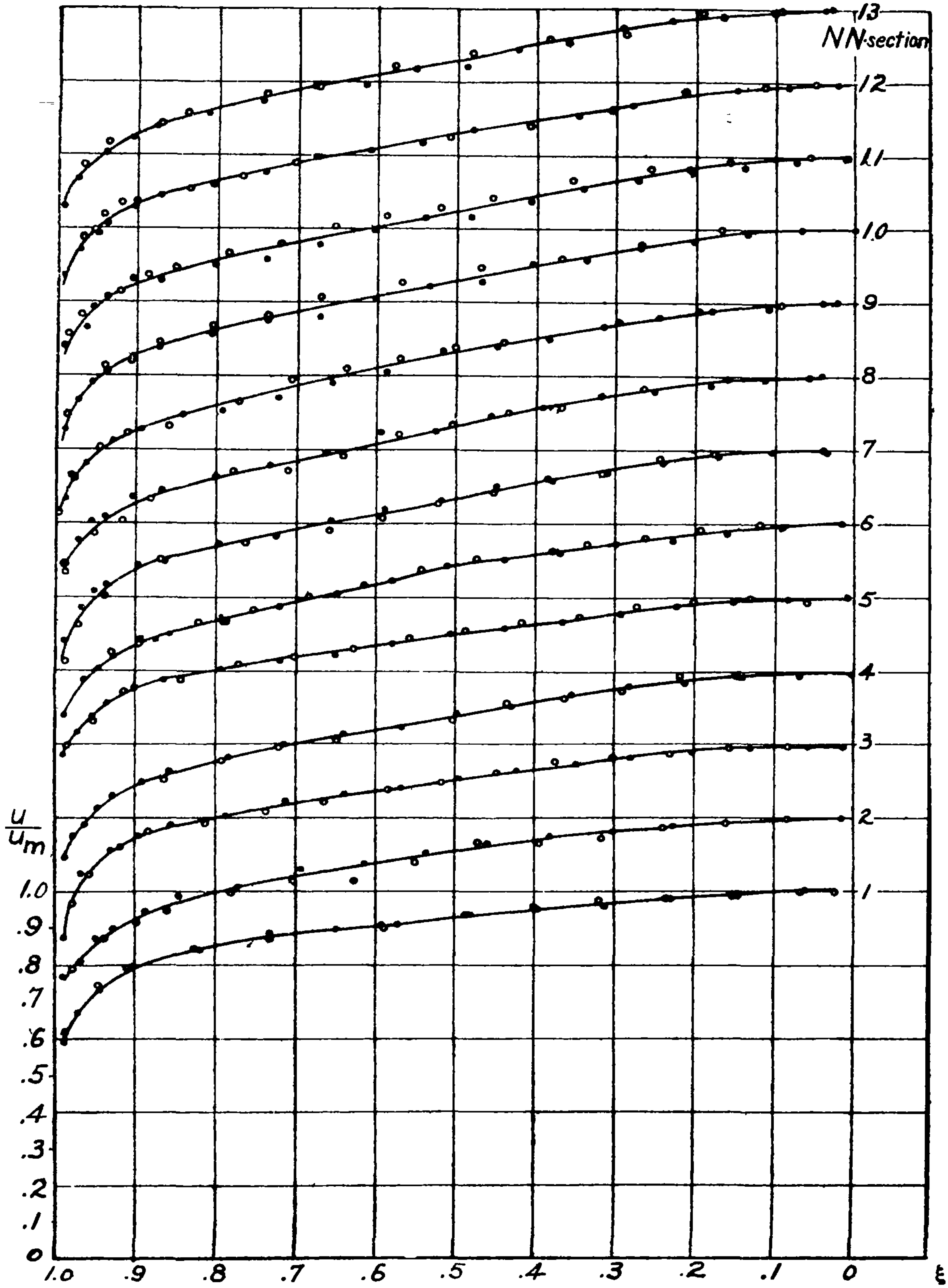


Figure 4.- Cross-sectional velocity distribution for 1° diffuser.

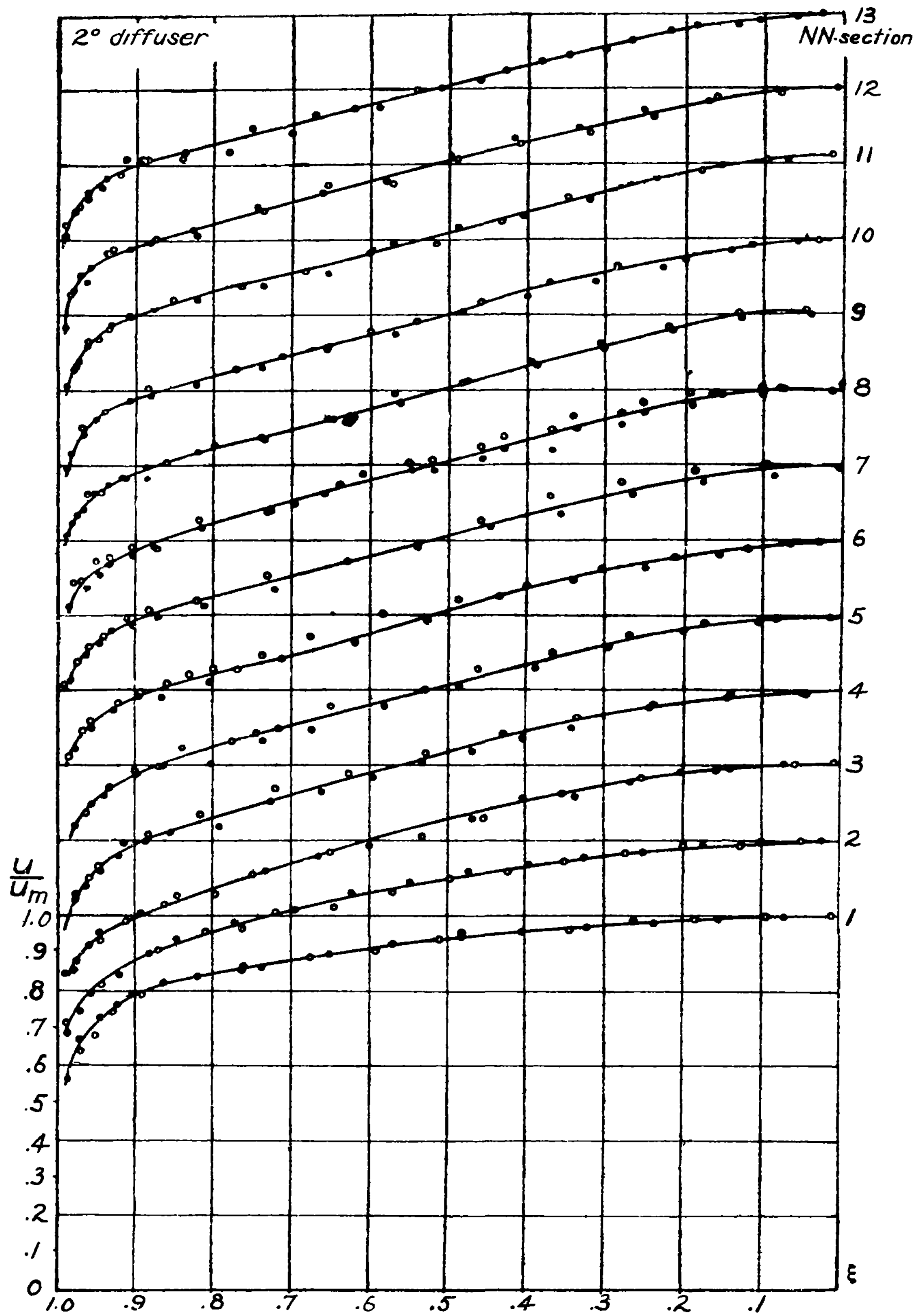


Figure 5.- Cross-sectional velocity distribution for 2° diffuser.

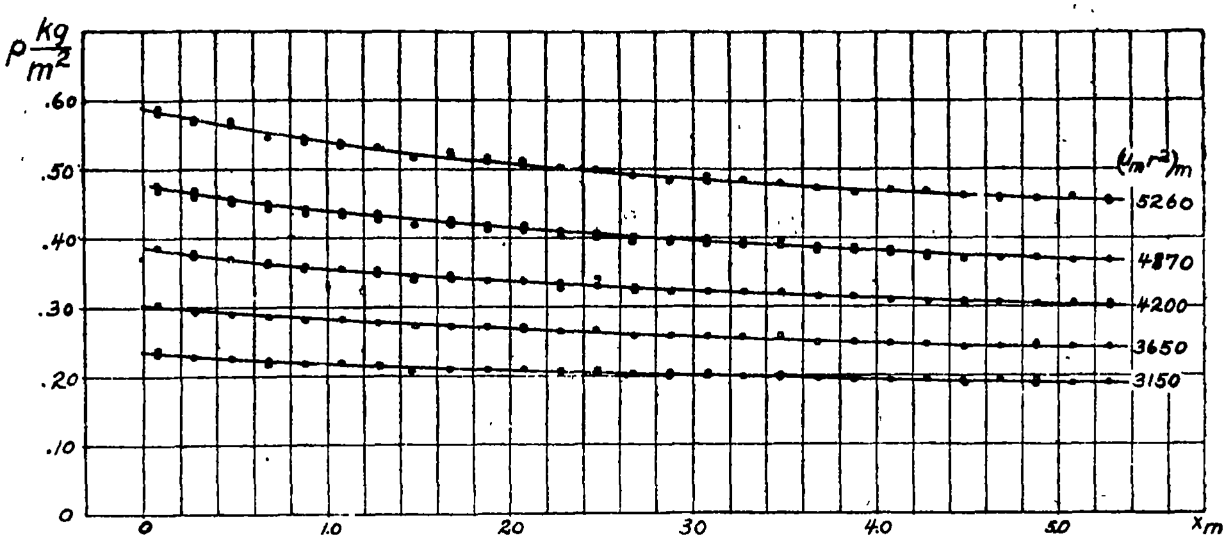


Figure 6.- Static pressure distribution along 1° diffuser.

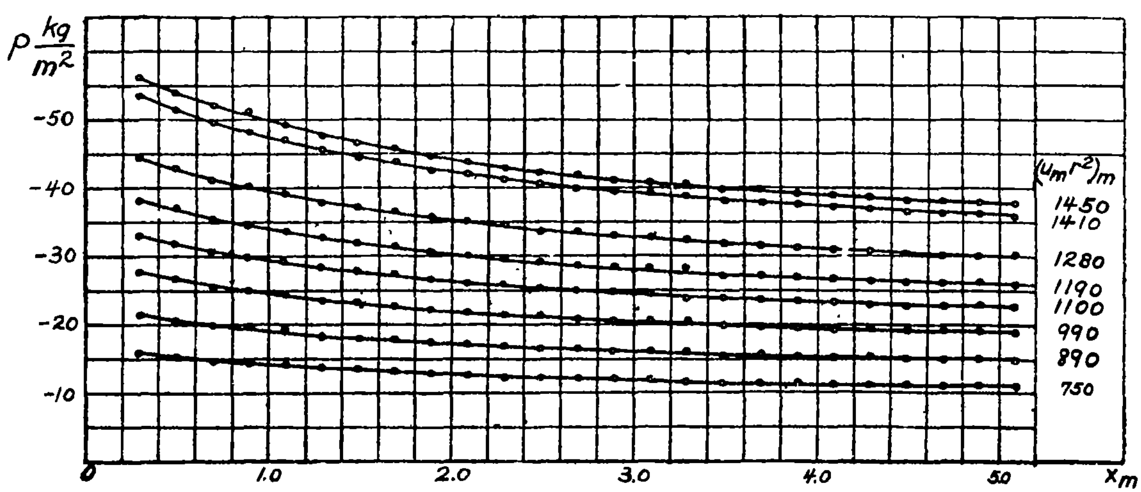


Figure 7.- Static pressure distribution along 2° diffuser.

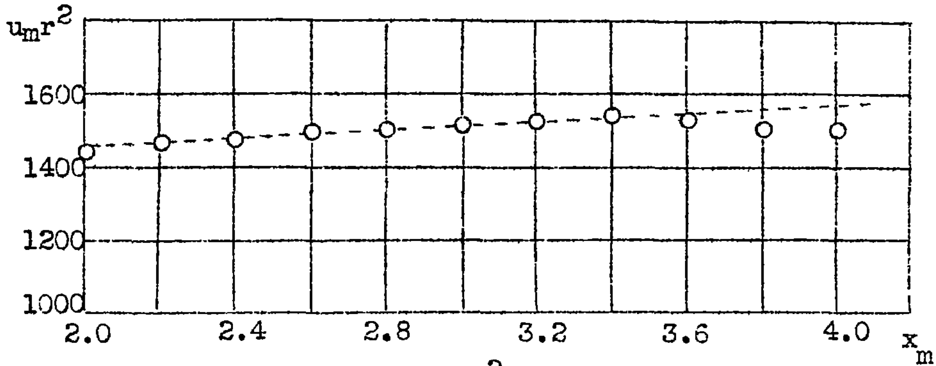


Figure 9.- Variation of u_mr^2 along axis of 2° diffuser for maximum discharge rate.

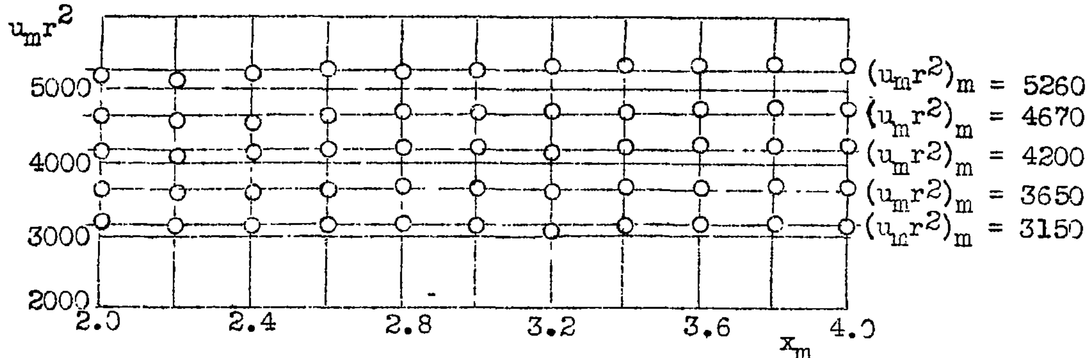


Figure 10.- Variation of u_mr^2 along axis of 1° diffuser for various discharge rates.

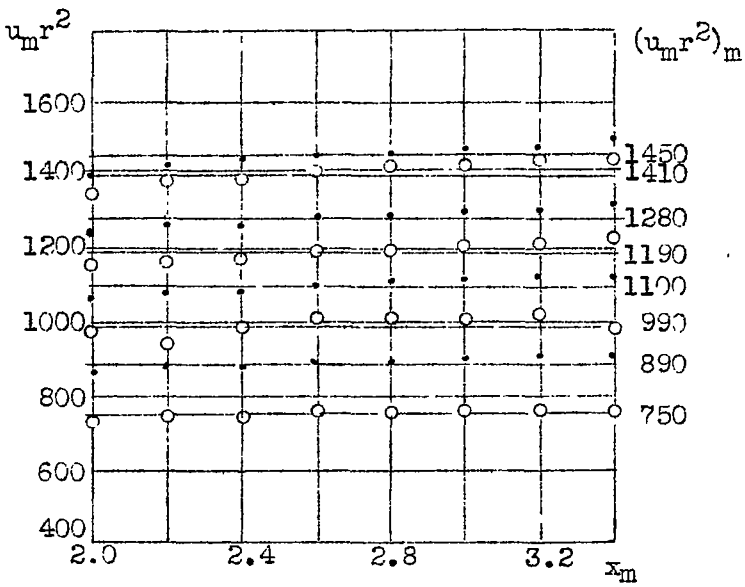


Figure 11.- Variation of u_mr^2 along axis of 2° diffuser for various discharge rates.

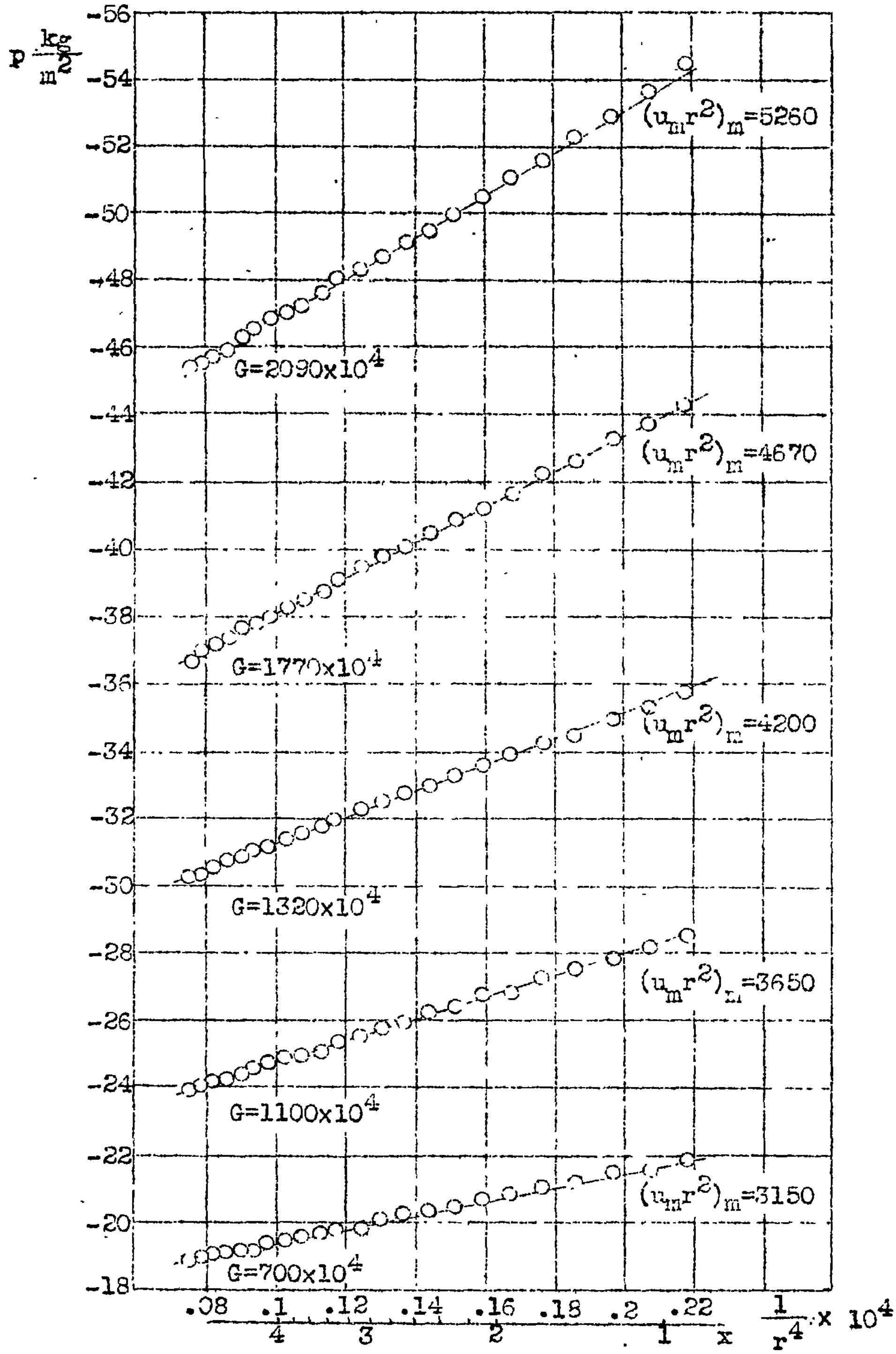


Figure 12.- Pressure distribution along 1° diffuser, $p = f(1/r^4)$.

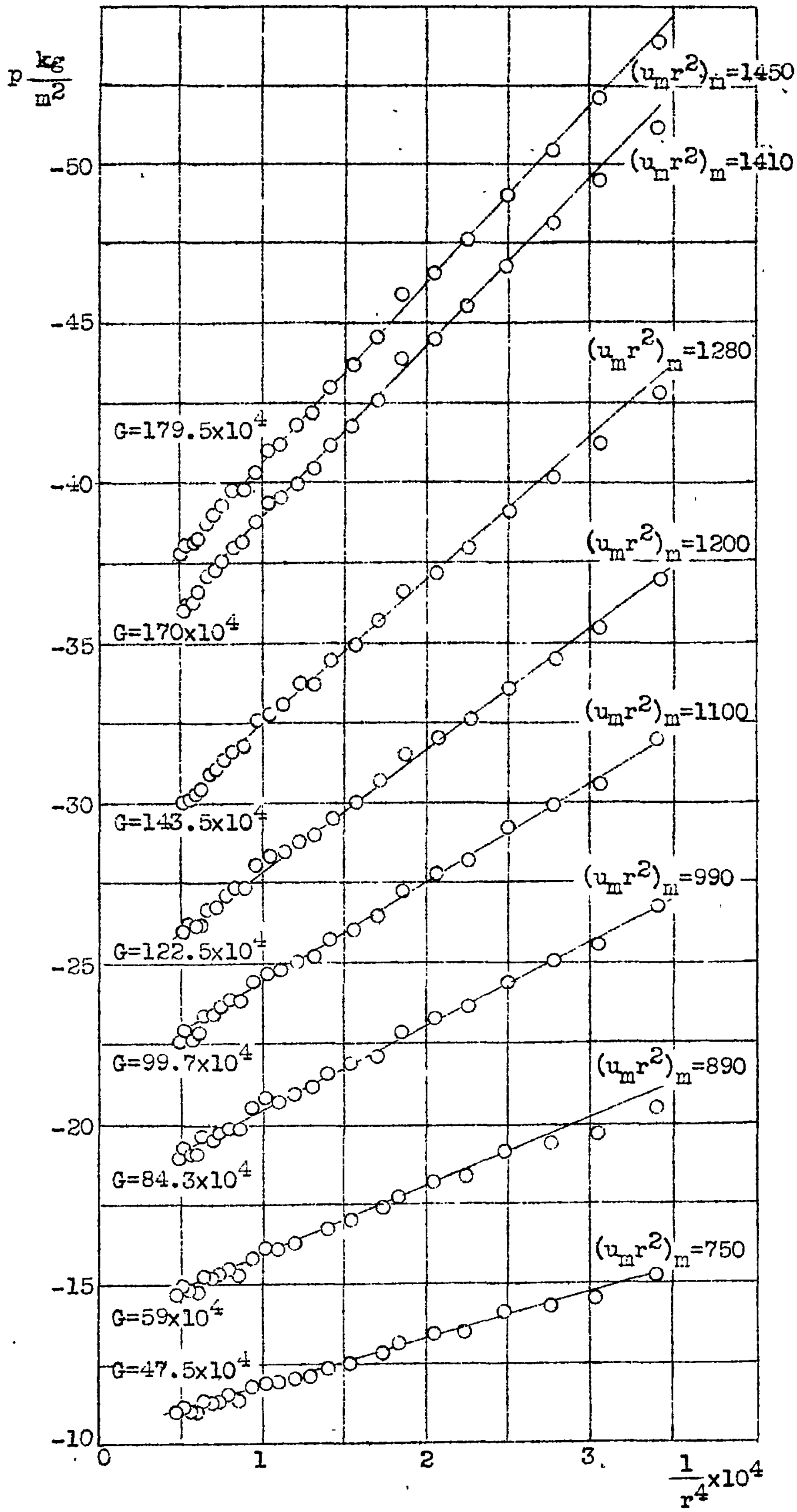


Figure 13.- Pressure distribution along 2^o diffuser, $p = f(1/r^4)$.

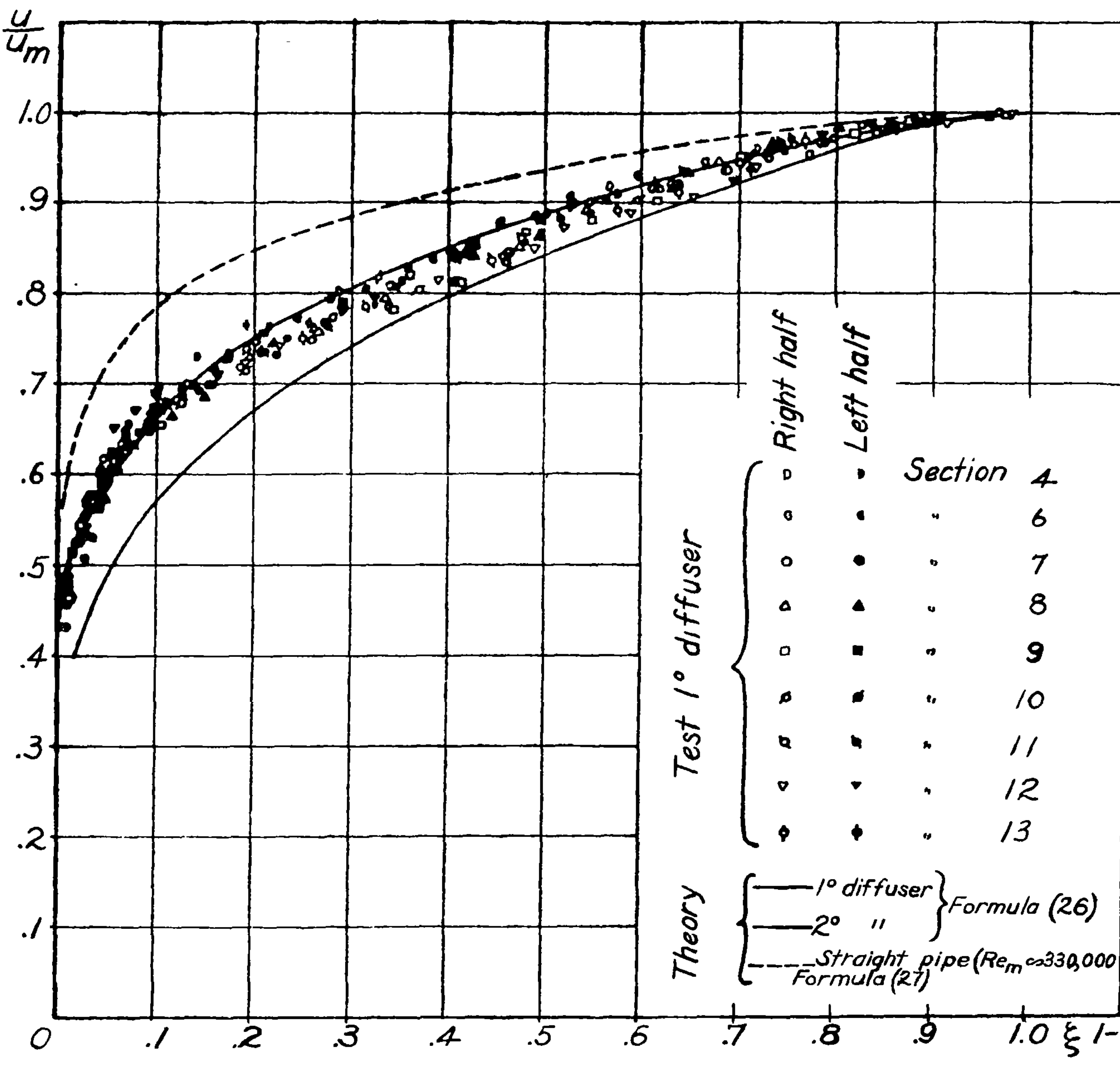


Figure 14.- Comparison of theoretical with experimental velocity profiles for 1° diffuser.

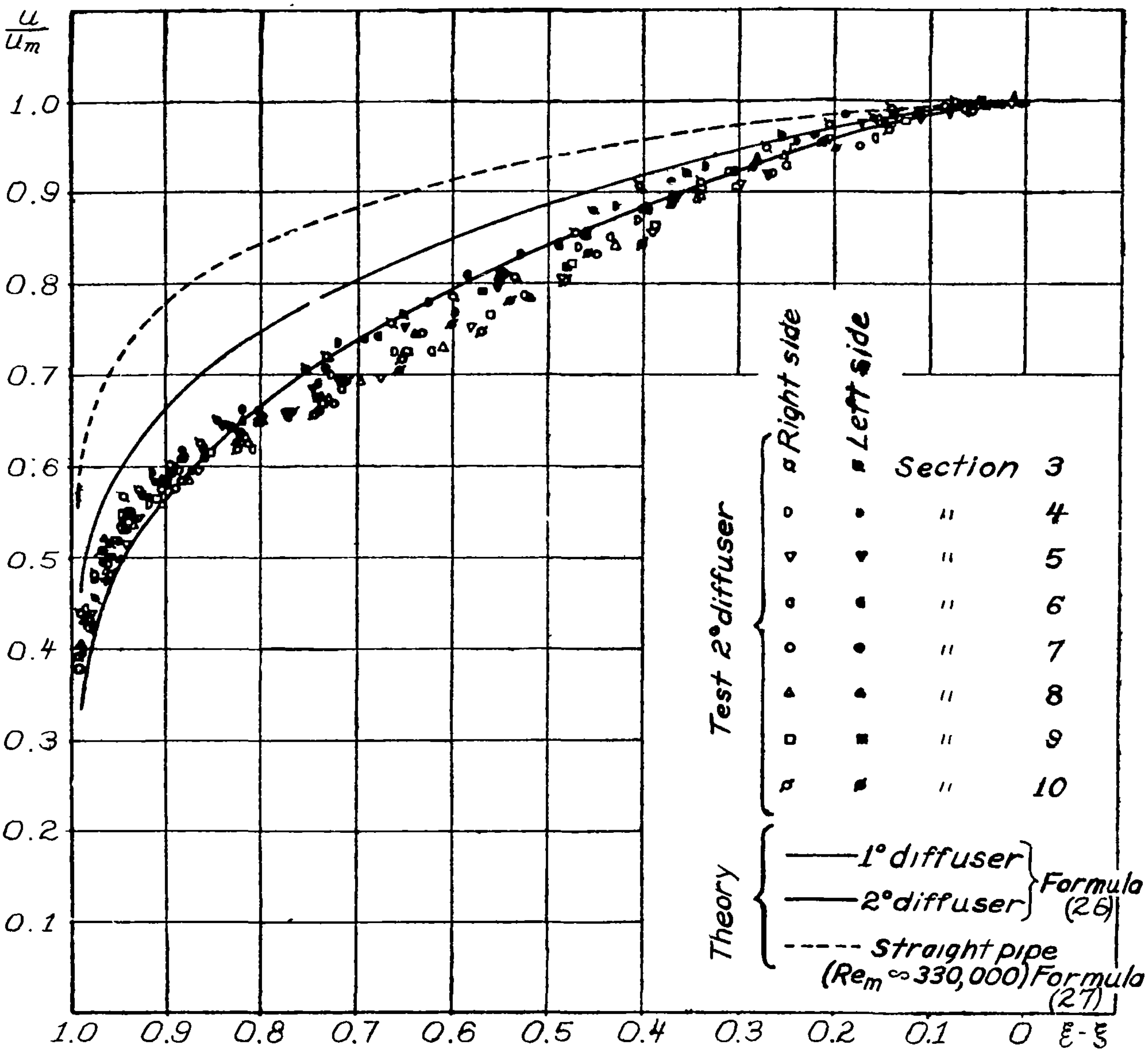


Figure 15.- Comparison of theoretical with experimental velocity profiles for 2° diffuser.

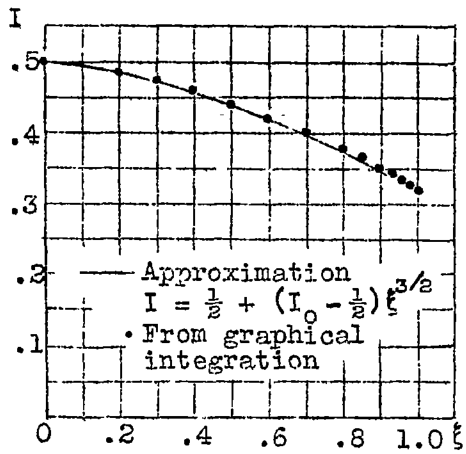
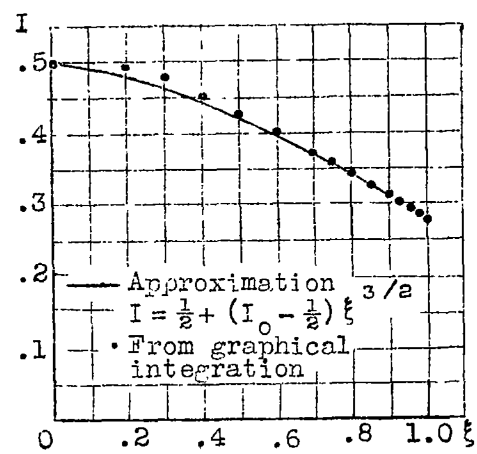


Figure 16.- Comparison of the approximation for the inertia integral $I = \frac{1}{\xi^2}\int_0^\xi \left(\frac{u}{u_m}\right)^2 \xi d\xi$ with the results of graphical integration from the theoretical velocity distribution for the 1° diffuser.

Figure 17.- Comparison of the approximation for the inertia integral $I = \frac{1}{\xi^2}\int_0^\xi \left(\frac{u}{u_m}\right)^2 \xi d\xi$ with the results of graphical integration from the theoretical velocity distribution for the 2° diffuser.



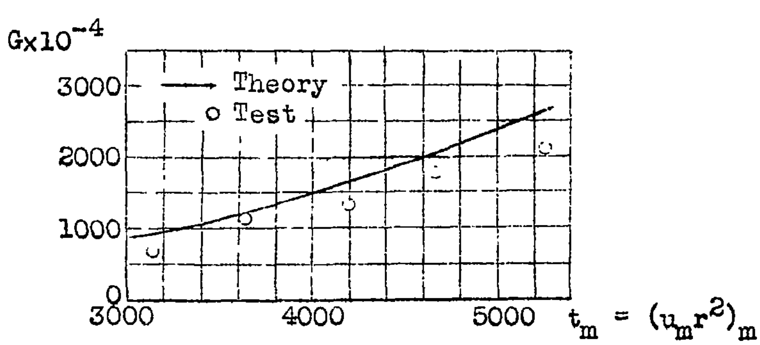


Figure 19.- Dependence of $G = -\frac{4}{\rho}\frac{dp}{d(r^{-4})}$ on $t_m = (u_m r^2)_m$ for the 1° diffuser.

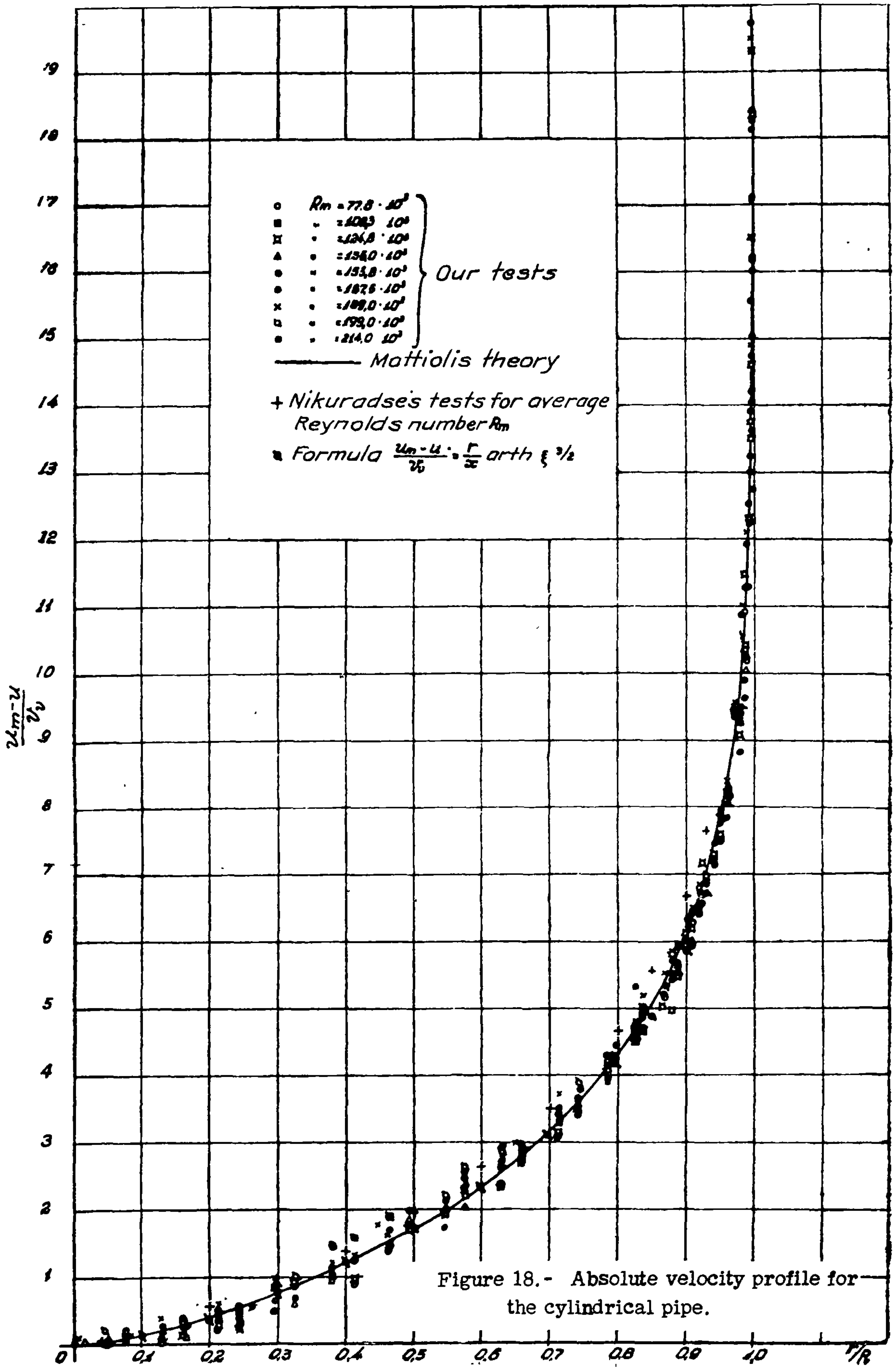


Figure 18.- Absolute velocity profile for the cylindrical pipe.

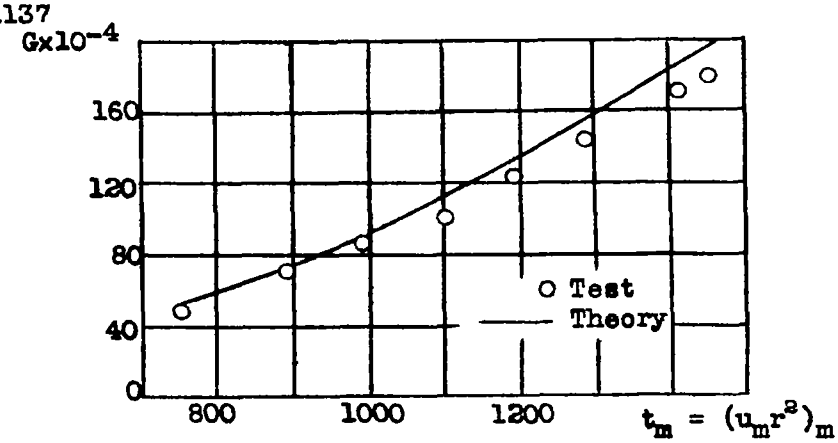


Figure 20.- Dependence of $G = -\frac{4}{\rho}\frac{dp}{d(r^{-4})}$ on $t_m = (u_m r^2)_m$ for the 2° diffuser.

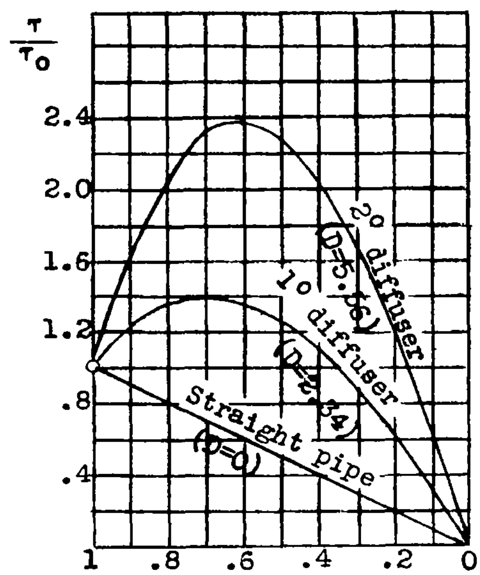


Figure 21.- Friction distribution across the straight pipe and diffusers according to the formula $\frac{\tau}{\tau_0} = \xi\,[D(1-\xi^{3/2}) + 1]$.

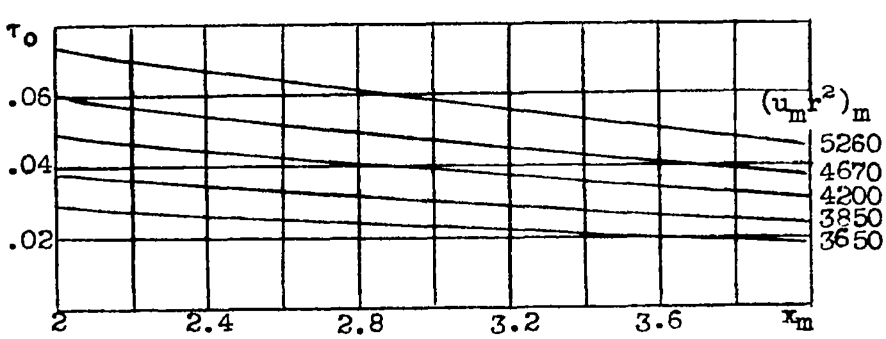


Figure 22.- Distribution of the frictional shear along the wall of the 1° diffuser for various values of $(u_m r^2)_m$.

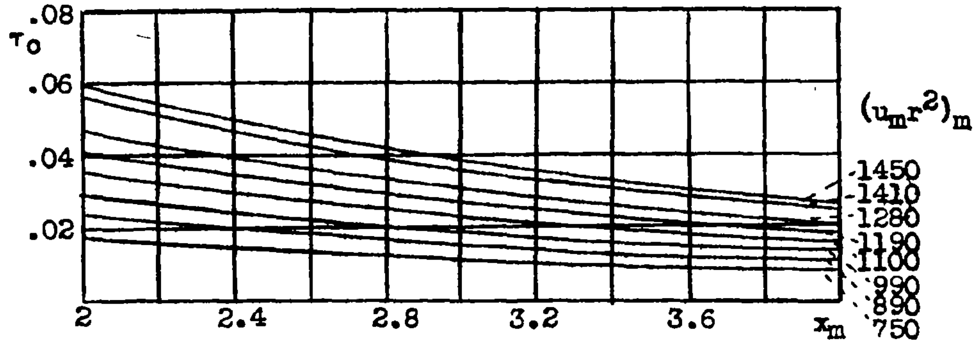


Figure 23.- Distribution of the frictional shear along the wall of the 2° diffuser for various values of $(u_m r^2)_m$.

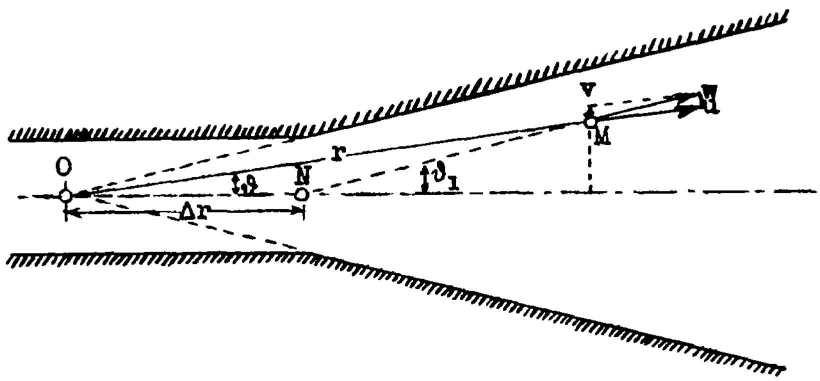


Figure 24.- Construction of "fictitious source" in the diffuser.

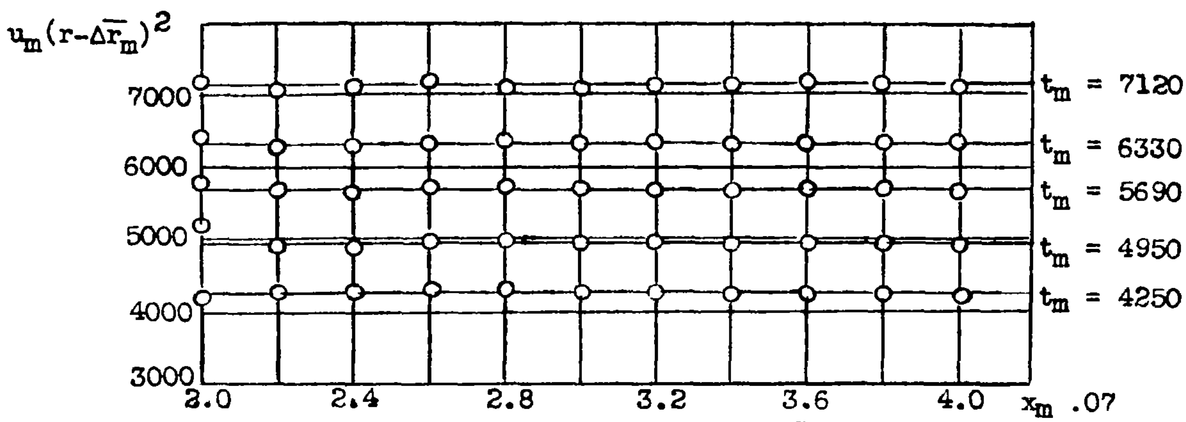


Figure 25.- Plot of product $u_m(r-\overline{\Delta r}_m)^2$ as a function of x for $\overline{\Delta r}_m = -2.75$ m for 1° diffuser.

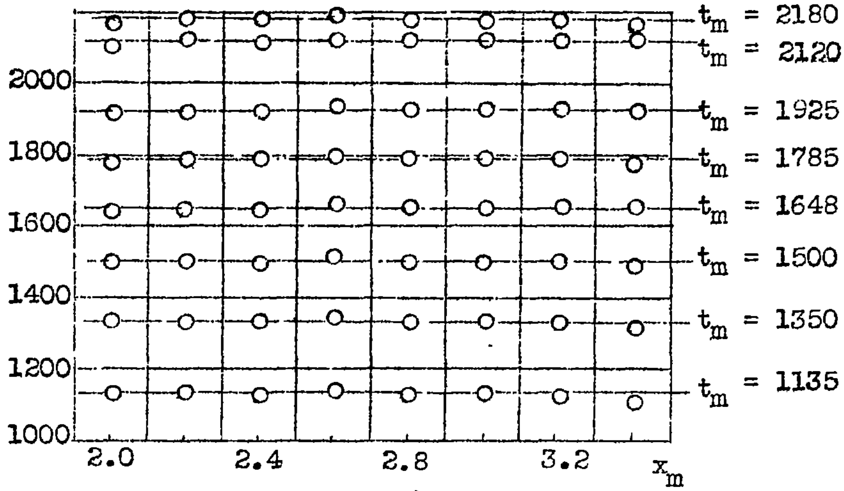


Figure 26.- Plot of product $u_m\,(r-\overline{\Delta r}_m)^2$ as a function of x for $\overline{\Delta r}_m = -\,2.27$ m for 2° diffuser.

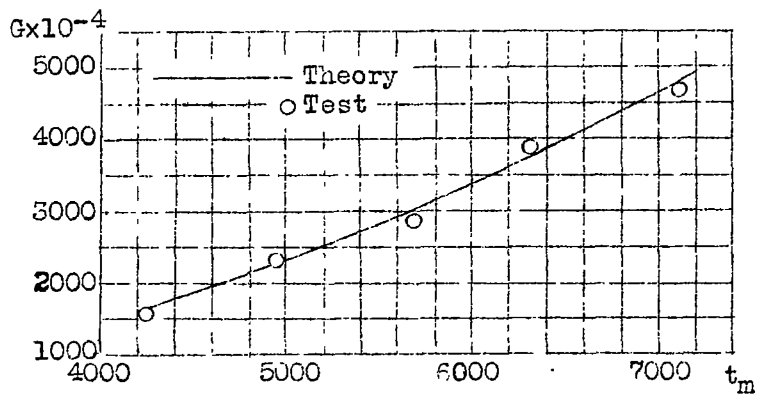


Figure 29.- Dependence of $G = -\frac{4}{\rho}\,\frac{dp}{d\left[(r-\overline{\Delta r}_m)^{-4}\right]}$ on $t_m = u_m(r-\overline{\Delta r}_m)^2$ for 1° diffuser for $\overline{\Delta r}_m = -\,2.75$ m.

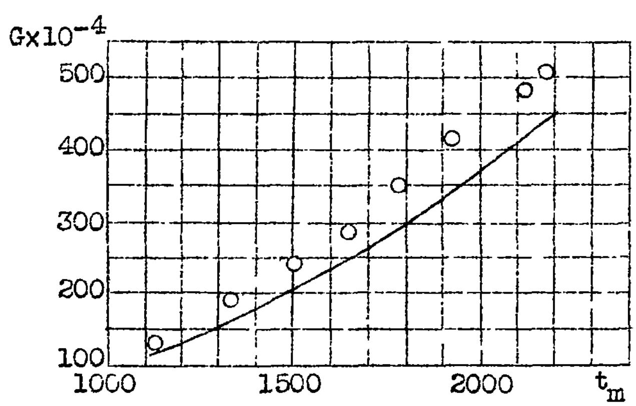


Figure 30.- Dependence of $G = -\frac{4}{\rho}\,\frac{dp}{d\left[(r-\overline{\Delta r}_m)^{-4}\right]}$ on $t_m = u_m(r-\overline{\Delta r}_m)^2$ for 2° diffuser for $\overline{\Delta r}_m = -\,2.27$ m.

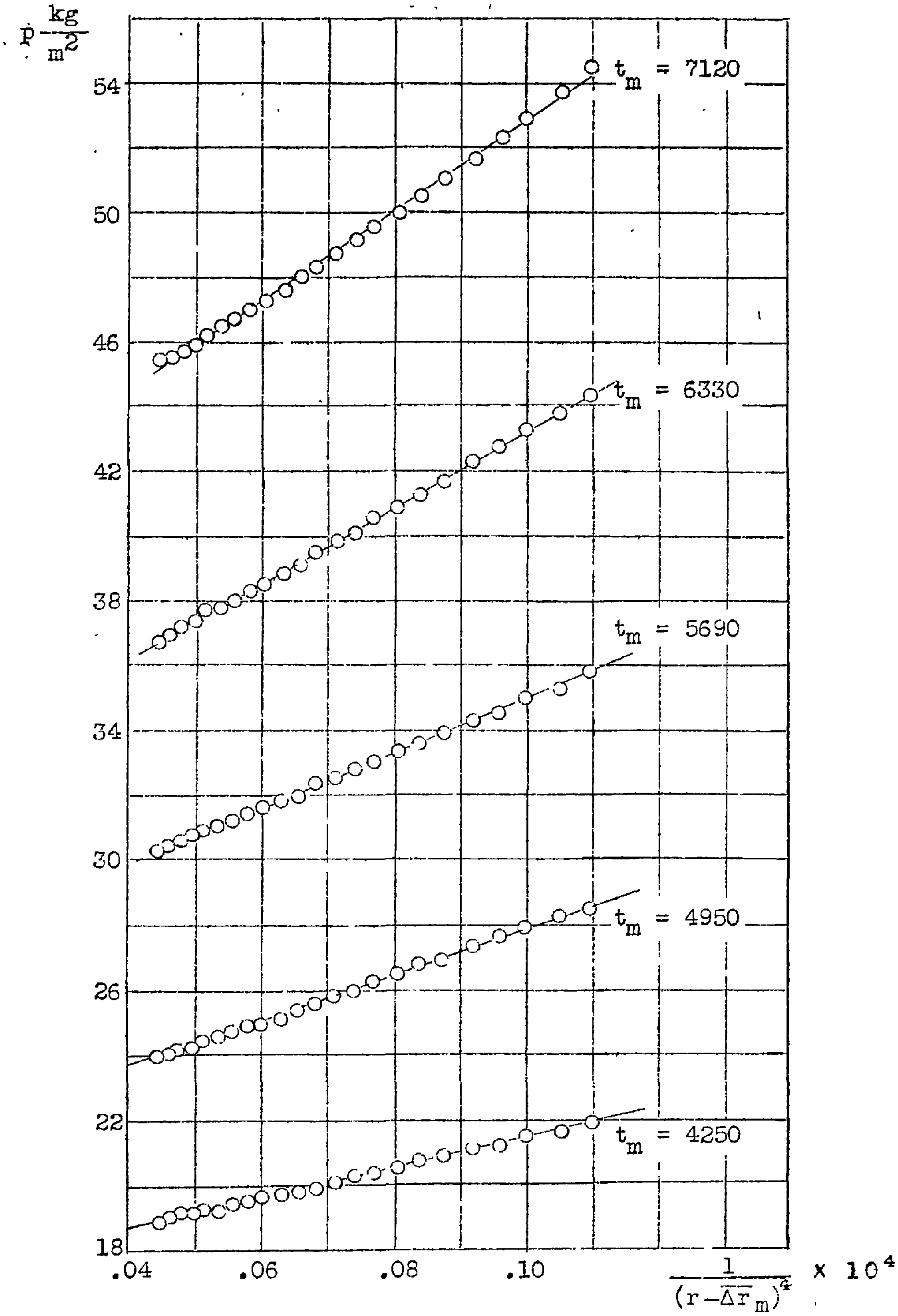


Figure 27.- Pressure distribution along 1° diffuser,

$$p = f\left[\frac{1}{(r-\overline{\Delta r}_m)^4}\right] \text{ for } \overline{\Delta r}_m = -2.75 \text{ m.}$$

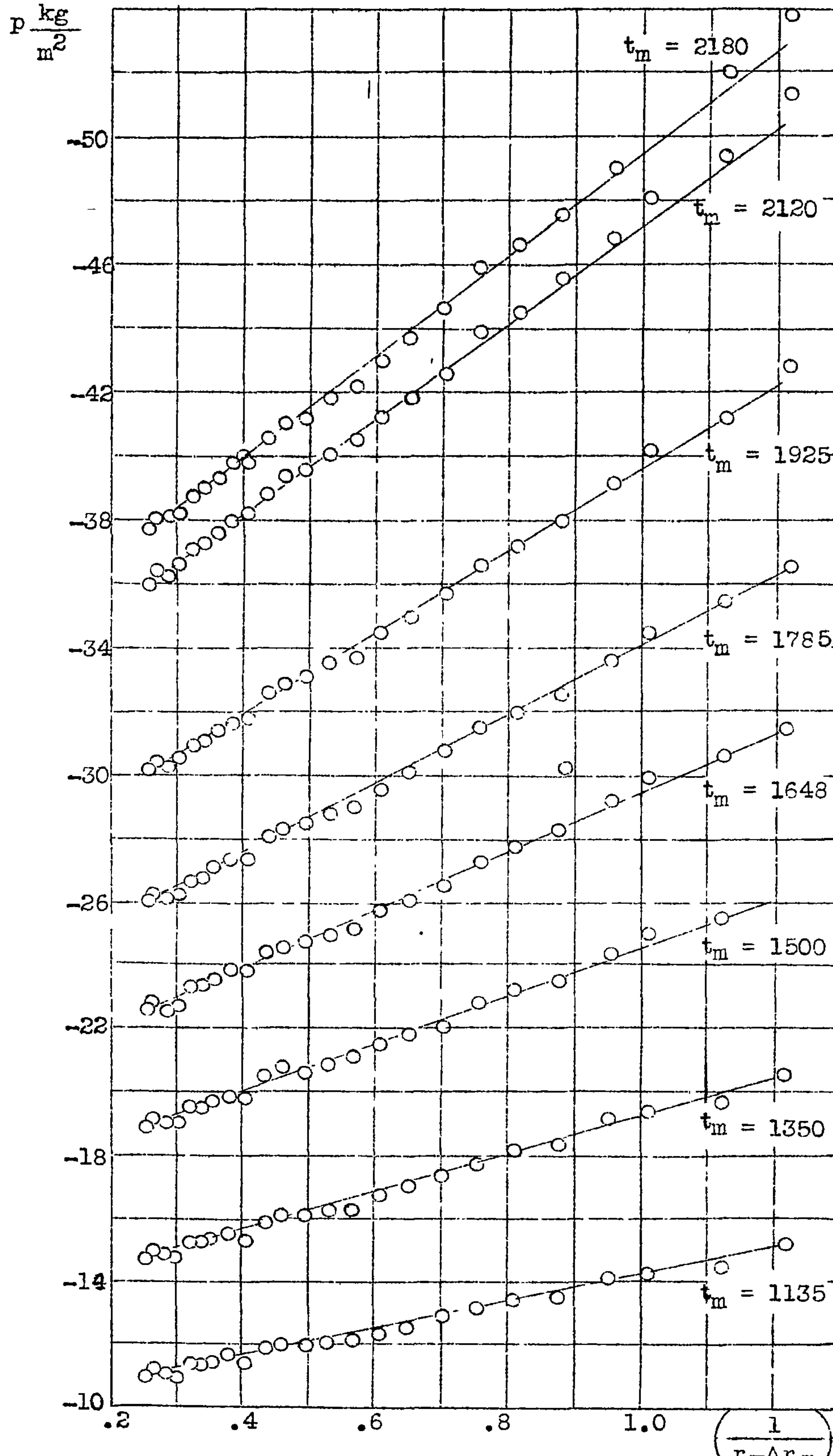


Figure 28.- Pressure distribution along 2° diffuser,

$$p = f\left[\frac{1}{(r - \overline{\Delta r}_m)^4}\right] \text{ for } \overline{\Delta r}_m = -22.7 \text{ m.}$$